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Abstract

In this paper we study the asymptotic behaviour of a sequence of two-dimensional linear elasticity problems with equicoercive elasticity tensors. Assuming the sequence of tensors is bounded in L^1, we obtain a compactness result extending to the elasticity the div-curl approach of [12] for the conduction. In the periodic case this compactness result is refined replacing the L^1-boundedness by a less restrictive condition involving the oscillations period. We also build a sequence of isotropic elasticity problems with L^1-unbounded Lamé's coefficients, which converges to a second gradient limit problem. This loss of compactness shows a gap in the limit behaviour between the very stiff problems of elasticity and those of conduction. Indeed, in the conduction case a compactness result was proved in [13] without assuming any bound from above for the conductivities.

Résumé

Dans cet article, on étudie le comportement asymptotique d'une suite de problèmes d'élasticité linéaire bidimensionnelle avec des tenseurs d'élasticité équi-coercifs. En supposant que la suite des tenseurs est bornée dans L^1, on établit un résultat de compacité qui étend à l'élasticité l'approche div-rot de [12] pour la conduction. Dans le cas périodique, on obtient un raffinement de ce résultat en remplaçant la borne L^1 des tenseurs par une condition moins restrictive faisant intervenir la période des oscillations. On construit également une suite de problèmes d'élasticité isotrope avec des coefficients de Lamé non bornés dans L^1, qui converge vers un problème limite avec un second gradient. Cette perte de compacité montre une différence notable de comportement limite entre les problèmes très raides d'élasticité et ceux de conduction. Dans ce dernier cas, en effet, un résultat de compacité a été prouvé dans [13] sans aucune hypothèse sur la borne supérieure des coefficients de conductivité.

Key words: Homogenization, H-convergence, Γ-convergence, elasticity.

1 Introduction

This paper deals with the asymptotic behaviour of two-dimensional linear elasticity problems with general sequences of equicoercive but non-uniformly bounded tensors. This contribution takes place in the larger topic of the homogenization of elliptic problems with high-contrast coefficients. Khruslov [18], [21] (see also the recent book [22]) was one of the first authors to deduce from high-contrast conductivities various types of limit behaviours: vector-valued problems, nonlocal and memory effects. His pioneer works have been extended in different directions (see e.g. [23], [8], [3], [14], [15], [9], and [10]). In elasticity the fiber reinforcement principle, which was first introduced by Khruslov in conduction to derive nonlocal effects, has been used successfully by several authors to obtain degenerate constitutive laws like second gradient materials [26] and nonlocal effects [4], [5]. However, there is a fundamental difference between the conduction and the elasticity. In conduction, by virtue of the truncation principle Mosco [23] proved that the limit energy associated with the homogenized problem satisfies the Beurling-Deny representation formula [7] of the Dirichlet forms. On the contrary, Seppecher and the second author [16] showed there is no such a restriction since remarkably any lower semicontinuous objective (i.e. vanishing for rigid motions) quadratic functional of displacements is the limit of a suitable sequence of isotropic elasticity energies with high-contrast Lamé's coefficients.

The previous results are three-dimensional since they are based on fiber reinforced structures. In dimension two the situation is completely different at least in conduction. Indeed, Casado-Díaz and the first author recently proved in [11] (for the periodic case), [12], [13], that dimension two prevents from any degeneracy, like nonlocal effects, which may arise in dimension three. These results can be regarded as a *compactness* result in the following sense: the class of conduction equations with equicoercive conductivities is closed for the L^2-strong convergence of the potentials. More precisely, for any bounded open subset Ω of $\mathbb{R}^2$, the solution $u_\varepsilon \in \mathrm{H}^1_0(\Omega)$ of the conduction problem $-\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f$ in $\mathcal{D}'(\Omega)$, with equicoercive conductivity A^ε, strongly converges in $\mathrm{L}^2(\Omega)$, up to a subsequence of ε, to the solution of a limit conduction problem of the same nature. From the energy point of view, any sequence of equicoercive diffusion energies, *i.e.,* of the type $\int_\Omega A^\varepsilon \nabla u \cdot \nabla u \, dx$, Γ-converges (see the definition in Theorem 3) for the strong topology of $\mathrm{L}^2(\Omega)$ to a diffusion energy. In other words, the nature of the original problem is preserved through the homogenization process in dimension two contrary to dimension three or greater.

The works [11], [12], [13] are based on two different approaches. On the one hand, assuming that the sequence of conductivities is equicoercive and bounded in L^1, Casado-Díaz and Briane proved in [11] and [12] extensions of the classical div-curl lemma of Murat-Tartar [24], which allowed them to deduce several compactness results in the sense above. In particular, the Murat-Tartar H-convergence [28], which holds for equicoercive and uniformly bounded conductivity coefficients, is generalized in [12] to the case of non-uniformly bounded coefficients. On the other hand, they improved in [13] the compactness result of [12] by getting rid of the L^1-boundedness assumption. The second approach is based both on a capacitary estimate and the maximum principle. Therefore, it cannot be extended to the elasticity case.

Now, the natural question is to know if the two-dimensional results of [11], [12], [13] still hold true in elasticity. The aim of the paper is to answer this question. To this end, we consider a general sequence of linearized elasticity problems posed in a bounded open subset Ω of $\mathbb{R}^2$, with a tensor-valued function $\boldsymbol{A}^\varepsilon$, $\varepsilon > 0$, defined on Ω, a volume force $\boldsymbol{f} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$, and whose displacement solution $\boldsymbol{u}^\varepsilon$ vanishes on $\partial\Omega$. Assuming the equicoercivity and the $L^1(\Omega)$-boundedness of $\boldsymbol{A}^\varepsilon$ (see (2.1)) we obtain (see Theorem 1) a compactness result by adapting the div-curl approach of [12] to elasticity. More precisely, we prove there exists a subsequence of ε, still denoted by ε, and a coercive and bounded tensor $\boldsymbol{A}^*$ such that, for any force $\boldsymbol{f} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$, the sequence $\boldsymbol{u}^\varepsilon$ weakly converges in $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ to the solution of the homogenized elasticity problem with tensor $\boldsymbol{A}^*$.

Using a refinement of the div-curl lemma (see Lemma 2) we obtain an improvement of the above result in the periodic framework, *i.e.,* when $\boldsymbol{A}^\varepsilon$ is ε-periodic (see Theorem 2). In this case, we can replace the $\mathrm{L}^1(\Omega)$-boundedness assumption by a less restrictive control of the L^1-

norm involving the square of the period ε. Therefore, the div-curl approach of [11] and [12] extends without restriction to the elasticity case.

On the contrary, we prove (see Theorem 3) that the result of [13] does not hold in elasticity, when the coefficients are not bounded in $\mathrm{L}^1(\Omega)$. Indeed, adapting the fiber reinforcement microstructure of [26] to dimension two, we build an equicoercive sequence of ε-periodic isotropic elasticity tensors, with very high Lamé's coefficients, such that the corresponding sequence of displacements $\boldsymbol{u}^\varepsilon$ weakly converges to the solution of an elasticity problem with fourth-order derivatives (see Theorem 4 and Corollary 1). In fact, similarly to [26] we adopt the (equivalent) energy point of view using a Γ-convergence approach (see, *e.g.* [17]). Therefore, there is a loss of compactness in elasticity in the sense that we provide a sequence of two-dimensional elasticity problems the limit of which takes us out of the class due to the appearance of a second gradient term. We can also conclude that the compactness result of [13] is definitely based on scalar elliptic ingredients like the maximum principle.

To sum up, the asymptotic behaviours of the conduction and elasticity equicoercive problems with very stiff coefficients agree when the coefficients are assumed to be bounded in L^1. Without this assumption and contrary to the conduction case of [13], sequences of two-dimensional elasticity problems with very stiff coefficients can induce extra terms in the limit problem (see Theorems 3, 4, Corollary 1 and Remarks 3, 4, 5 detailing the gap between conduction and elasticity).

The structure of the paper is the following. In Section 2 we set up the notations and state the main results. Section 3 is devoted to the proofs of the results. We first prove a compactness result (Theorem 1) adapting the techniques of [12] to the elasticity setting. Then we prove a div-curl result (Lemma 2) in the periodic case. Once this div-curl result is proved, we proceed with the proof of Theorem 2. The last subsection of Section 3 is devoted to the demonstration of a result (Theorem 3) which shows that, in dimension two, there is a gap between the asymptotic behaviour of conduction problems and the elasticity ones.

2 Main results

2.1 Notations and definitions

- Ω is a bounded connected open subset of $\mathbb{R}^2$ with a Lipschitz boundary. The unit square $(0,1)^2$ of $\mathbb{R}^2$ is denoted by Y.
- For any subset ω of Ω, we denote by $\omega \Subset \Omega$ the inclusion $\bar{\omega} \subset \Omega$, where $\bar{\omega}$ stands for the closure of ω in $\mathbb{R}^2$.
- The space of (2×2) real-valued symmetric matrices is denoted by $\mathbb{R}^{2\times 2}_s$. The identity matrix of $\mathbb{R}^{2\times 2}_s$ is denoted by $\boldsymbol{I}_2$, while the identity fourth-order tensor is denoted by $\boldsymbol{I}_4$.
- The scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ of $\mathbb{R}^2$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, and the one of two matrices $\boldsymbol{\sigma}, \boldsymbol{\xi} \in \mathbb{R}^{2\times 2}$ is denoted by $\boldsymbol{\sigma} : \boldsymbol{\xi} = \mathrm{Tr}(\boldsymbol{\sigma}^t \boldsymbol{\xi})$, where $\boldsymbol{\sigma}^t$ is the transpose of $\boldsymbol{\sigma}$ and $\mathrm{Tr}(\boldsymbol{\sigma})$ its trace. We denote the norm of a vector $\boldsymbol{u} \in \mathbb{R}^2$ by $|\boldsymbol{u}|$ and the one of a matrix $\boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}$ by $\|\boldsymbol{\sigma}\|$.
- For any $\boldsymbol{\sigma}$ and $\boldsymbol{\xi}$ in $\mathbb{R}^{2\times 2}_s$ we denote by $\boldsymbol{\sigma} \otimes \boldsymbol{\xi}$ the fourth-order tensor the components of which are defined by
$$(\boldsymbol{\sigma} \otimes \boldsymbol{\xi})_{ijkl} := \sigma_{ij}\xi_{kl}.$$

- The gradient of a displacement $\boldsymbol{u} \in \mathbb{R}^2$ is the (2×2) matrix $\nabla \boldsymbol{u}$ the entries of which are defined by

$$(\nabla \boldsymbol{u})_{ij} := \frac{\partial u_i}{\partial x_j}.$$

  The divergence of a matrix $\boldsymbol{\sigma}$ is the vector $\mathrm{Div}\,(\boldsymbol{\sigma})$ the components of which are defined by

$$\left(\mathrm{Div}\,(\boldsymbol{\sigma})\right)_i := \frac{\partial \sigma_{ij}}{\partial x_j},$$

  where the Einstein summation convention over repeated indices is used.

- The symmetric part of the gradient of a displacement $\boldsymbol{u}$ is denoted by $\boldsymbol{e}(\boldsymbol{u})$ *i*.e.,

$$\boldsymbol{e}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right).$$

  Note that, for any symmetric fourth-order tensor $\boldsymbol{A}$, we have $\boldsymbol{A}\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{A}\nabla \boldsymbol{u}$. Therefore, we will use indifferently both expressions.

- The support of a function φ is denoted by supp φ. The space of infinitely differentiable functions with compact support in Ω is denoted by $\mathcal{D}(\Omega)$.

- We denote by $C_0(\Omega)$ the space of continuous functions on $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$ of Ω, and by $\mathcal{M}(\Omega)$ the set of Radon measures on Ω. A sequence (μ_ε) in $\mathcal{M}(\Omega)$ is said to weakly $*$ converge to a measure μ if

$$\int_\Omega \varphi \mu_\varepsilon(dx) \underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega \varphi \mu(dx), \quad \text{for any } \varphi \in C_0(\Omega).$$

- We denote by $\mathrm{H}^1(\Omega, \mathbb{R}^2)$ the usual Sobolev space, endowed with its standard norm

$$\|\boldsymbol{u}\|_{\mathrm{H}^1(\Omega,\mathbb{R}^2)} := \left(\int_\Omega |\boldsymbol{u}(x)|^2\,dx + \int_\Omega |\nabla \boldsymbol{u}(x)|^2\,dx\right)^{1/2}.$$

- The space of Y-periodic functions which belong to $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^2)$ (resp. $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2)$) is denoted by $\mathrm{L}^p_\#(Y)$ (resp. $\mathrm{H}^1_\#(Y)$).

- We denote by $|\omega|$ the Lebesgue measure of any Borel subset $\omega \subset \Omega$, and by

$$\fint_\omega u\,dx := \frac{1}{|\omega|}\int_\omega u\,dx$$

  the average-value of any function $u \in \mathrm{L}^1(\omega)$.

- We denote by $\mathbf{1}_\omega$ the characteristic function of the set ω.

- $O(\varepsilon)$ denotes a term bounded by a constant times ε.

- Throughout the paper, the letter c denotes a positive constant whose value is not given explicitly and that may vary from line to line.

Let $(\boldsymbol{A}^\varepsilon)$ be a sequence of symmetric fourth-order tensor-valued functions satisfying

$$\begin{cases} \boldsymbol{A}^\varepsilon(x)\boldsymbol{\xi}:\boldsymbol{\xi} & \geq & \alpha\,\|\boldsymbol{\xi}\|^2, \\ (\boldsymbol{A}^\varepsilon(x))^{-1}\,\boldsymbol{\xi}:\boldsymbol{\xi} & \geq & (\beta_\varepsilon(x))^{-1}\|\boldsymbol{\xi}\|^2 \end{cases} \quad \text{a.e. } x\in\Omega,\ \forall\,\boldsymbol{\xi}\in\mathbb{R}^{2\times 2}_s, \tag{2.1}$$

for some positive constant α and some sequence (β_ε) in $\mathrm{L}^1(Y)$. Note that (2.1) implies that $\beta_\varepsilon(x)\geq\alpha$, a.e. $x\in\Omega$.

Example 1 In the particular case of isotropic elastic materials, the tensor $\boldsymbol{A}^\varepsilon$ is determined by the Lamé coefficients λ_ε and μ_ε as follows:

$$\boldsymbol{A}^\varepsilon(x)=2\mu_\varepsilon(x)\boldsymbol{I}_4+\lambda_\varepsilon(x)\boldsymbol{I}_2\otimes\boldsymbol{I}_2 \qquad \text{a.e. } x\in\Omega,$$

or equivalently,

$$\boldsymbol{A}^\varepsilon(x)\boldsymbol{\xi}=2\mu_\varepsilon(x)\boldsymbol{\xi}+\lambda_\varepsilon(x)\,\mathrm{Tr}(\boldsymbol{\xi})\boldsymbol{I}_2 \qquad \text{a.e. } x\in\Omega, \quad \forall\,\boldsymbol{\xi}\in\mathbb{R}^{2\times 2}_s.$$

Then, condition (2.1) is equivalent to

$$\alpha\leq 2\min\left(\mu_\varepsilon,\lambda_\varepsilon+\mu_\varepsilon\right) \quad \text{and} \quad \beta_\varepsilon\geq 2\max\left(\mu_\varepsilon,\lambda_\varepsilon+\mu_\varepsilon\right).$$

Let $\boldsymbol{f}$ be an element of $\mathrm{H}^{-1}(\Omega,\mathbb{R}^2)$. Consider the sequence of elasticity problems

$$\begin{cases} -\mathrm{Div}\left(\boldsymbol{A}^\varepsilon\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\right) & = & \boldsymbol{f} & \text{in} & \Omega \\ \boldsymbol{u}^\varepsilon & = & 0 & \text{on} & \partial\Omega. \end{cases} \tag{2.2}$$

In the periodic case, $\boldsymbol{A}^\varepsilon$ and β_ε read as

$$\boldsymbol{A}^\varepsilon(x):=\boldsymbol{A}^\varepsilon_\sharp\left(\tfrac{x}{\varepsilon}\right) \qquad \text{and} \qquad \beta_\varepsilon(x):=\beta^\sharp_\varepsilon\left(\tfrac{x}{\varepsilon}\right) \quad \text{a.e. } x\in\Omega,$$

where $\boldsymbol{A}^\varepsilon_\sharp$ and $\beta^\sharp_\varepsilon$ are Y-periodic functions. It is known (see for instance, [6], [2] or [1]) that, for a fixed $\varepsilon>0$, the oscillating sequence $\left(\boldsymbol{A}^\varepsilon(\tfrac{x}{\delta})\right)$ induces, as δ tends to zero, the constant homogenized tensor $\boldsymbol{A}^\varepsilon_*$ defined by the following minimization

$$\boldsymbol{A}^\varepsilon_*\boldsymbol{\xi}:\boldsymbol{\xi}=\min\left\{\int_Y \boldsymbol{A}^\varepsilon(y)\left(\boldsymbol{\xi}+\boldsymbol{e}(\boldsymbol{\Psi})\right):\left(\boldsymbol{\xi}+\boldsymbol{e}(\boldsymbol{\Psi})\right)\,dy:\boldsymbol{\Psi}\in\mathrm{H}^1_\#(Y,\mathbb{R}^2)\right\},\ \forall\,\boldsymbol{\xi}\in\mathbb{R}^{2\times 2}_s. \tag{2.3}$$

2.2 Homogenization results

2.2.1 The non-periodic case

By drawing upon the div-curl approach developed in [12] we establish a compactness result in elasticity when the sequence of Hooke's laws is L^1-bounded and equicoercive. More precisely, we have the following result:

Theorem 1 *Let $(\boldsymbol{A}^\varepsilon)$ be a sequence of symmetric fourth-order tensor-valued functions satisfying (2.1). Suppose that there exists a function $\beta\in\mathrm{L}^\infty(\Omega)$ such that*

$$\beta_\varepsilon \rightharpoonup \beta \quad \textit{weakly in } \mathcal{M}(\bar\Omega)*. \tag{2.4}$$

Then, there exist a subsequence, still denoted by ε, and a symmetric fourth-order tensor $\boldsymbol{A}^*$ satisfying

$$\begin{cases} \boldsymbol{A}^*\boldsymbol{\xi}:\boldsymbol{\xi} & \geq & \alpha\,\|\boldsymbol{\xi}\|^2, \\ (\boldsymbol{A}^*)^{-1}\,\boldsymbol{\xi}:\boldsymbol{\xi} & \geq & (\|\beta\|_{\mathrm{L}^\infty(\Omega)})^{-1}\|\boldsymbol{\xi}\|^2 \end{cases} \quad \textit{a.e. } x\in\Omega,\ \forall\,\boldsymbol{\xi}\in\mathbb{R}^{2\times 2}_s, \tag{2.5}$$

such that, for any $\boldsymbol{f} \in \mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$*, the solution* $\boldsymbol{u}^\varepsilon$ *of the problem (2.2) satisfies*

$$\left\{ \begin{array}{rcll} \boldsymbol{u}^\varepsilon & \rightharpoonup & \boldsymbol{u} & \textit{weakly in } \mathrm{H}_0^1(\Omega, \mathbb{R}^2) \\ \boldsymbol{A}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) & \rightharpoonup & \boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u}) & \textit{weakly} * \textit{ in } \mathcal{M}(\Omega, \mathbb{R}_s^{2\times 2}), \end{array} \right. \tag{2.6}$$

where $\boldsymbol{u}$ *is the solution in* $\mathrm{H}_0^1(\Omega, \mathbb{R}^2)$ *of the elasticity problem*

$$\left\{ \begin{array}{rcll} -\mathrm{Div}\ (\boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u})) & = & \boldsymbol{f} & \textit{in} \quad \Omega \\ \boldsymbol{u} & = & \boldsymbol{0} & \textit{on} \quad \partial\Omega. \end{array} \right. \tag{2.7}$$

Remark 1 Let us notice that the Hooke's laws $\boldsymbol{A}^\varepsilon$ of Theorem 1 are supposed to be equicoercive but not necessarily uniformly bounded. Therefore, the result of Theorem 1 is an extension, in dimension two, of the H-convergence of Murat and Tartar [25] adapted to the elasticity by Francfort and Murat [19] (see Theorem 2.1, p. 313). The uniform upper boundedness is here replaced by a less restrictive condition, namely, the L^1-boundedness of the sequence $(\boldsymbol{A}^\varepsilon)$.

As in the classical H-convergence of [25] and the two-dimensional one of [12], the H-convergence result of Theorem 1 is based on a div-curl lemma which extends the classical one of Murat and Tartar [24] to the case where the sequences are not bounded in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$. The following lemma is an adaptation of the two-dimensional div-curl result of [12] (Theorem 2.1) a refinement of which is given in the periodic case of the next section (see Lemma 2).

Lemma 1 (Briane and Casado-Díaz [12]) *Let* $(\boldsymbol{A}^\varepsilon)$ *be a sequence of tensor-valued functions satisfying (2.1) and (2.4). Let* $(\boldsymbol{\sigma}^\varepsilon)$ *be a sequence in* $\mathrm{L}^2(\Omega, \mathbb{R}_s^{2\times 2})$ *and* $(\boldsymbol{v}^\varepsilon)$ *be a sequence weakly converging to some* $\boldsymbol{v}$ *in* $\mathrm{H}^1(\Omega, \mathbb{R}^2)$ *such that*

$$\int_\Omega (\boldsymbol{A}^\varepsilon)^{-1} \boldsymbol{\sigma}^\varepsilon : \boldsymbol{\sigma}^\varepsilon \, dx \le c \qquad \textit{and} \qquad \mathrm{Div}(\boldsymbol{\sigma}^\varepsilon) \textit{ is compact in } \mathrm{H}^{-1}(\Omega, \mathbb{R}^2).$$

Then, there exist a subsequence of ε*, still denoted by* ε*, and* $\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega, \mathbb{R}_s^{2\times 2})$ *such that the following convergences hold true*

$$\boldsymbol{\sigma}^\varepsilon \rightharpoonup \boldsymbol{\sigma} \quad \textit{weakly} * \ \textit{in } \mathcal{M}(\Omega, \mathbb{R}_s^{2\times 2}) \qquad \textit{and} \qquad \boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{\sigma} : \nabla \boldsymbol{v} \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.8}$$

This is a straightforward adaptation of the two-dimensional div-curl lemma of [12] (Theorem 2.1) applied to the k-th rows $\boldsymbol{\sigma}_k^\varepsilon$ and $\boldsymbol{v}_k^\varepsilon$ of $\boldsymbol{\sigma}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$ respectively, noting that

$$\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon = \boldsymbol{\sigma}_1^\varepsilon \cdot \nabla \boldsymbol{v}_1^\varepsilon + \boldsymbol{\sigma}_2^\varepsilon \cdot \nabla \boldsymbol{v}_2^\varepsilon.$$

□

2.2.2 The periodic case

The assumption (2.4) in Theorem 1 is a constraint which is not fulfilled as soon as the L^1-norm of β_ε is not bounded. However, if $\boldsymbol{A}^\varepsilon$ and β_ε are supposed to be periodic, then we can weaken assumption (2.4) allowing β^ε not to be bounded in $\mathrm{L}^1(\Omega)$. The proof of this result (Theorem 2) is based on the application of a result (Lemma 2) which is a refinement (thanks to the periodicity) of the div-curl lemma 1.

Lemma 2 *Let ω be a bounded connected open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Let $(\beta_\varepsilon^\sharp)$ be a sequence of Y-periodic positive functions satisfying (2.12). Consider (v^ε) in $\mathrm{H}^1(\omega)$ and (V^ε) in $\mathrm{L}^2_\#(Y)$ satisfying*

$$\int_\omega v^\varepsilon \, dx = 0 \qquad \text{and} \qquad \int_Y V^\varepsilon \, dy = 0, \tag{2.9}$$

$$\int_\omega (\beta_\varepsilon^\sharp)^{-1} \left(\frac{x}{\varepsilon}\right) |\nabla v^\varepsilon|^2 \, dx + \int_Y |V^\varepsilon|^2 \, dy \le c, \tag{2.10}$$

for some positive constant c. Then, the sequence $v^\varepsilon(\cdot)V^\varepsilon(\frac{\cdot}{\varepsilon})$ converges to zero in $\mathcal{D}'(\omega)$.

Remark 2 The improvement of the result of Lemma 2 in comparison with that of [12] consists in the fact that in the div-curl Lemma of [12] the sequence (β_ε) is supposed to be L^1-bounded, while in Lemma 2 the L^1-norm of $(\beta_\varepsilon^\sharp)$ may tend to infinity, as long as

$$\|\beta_\varepsilon^\sharp\|_{\mathrm{L}^1(Y)} \ll \varepsilon^{-2}. \tag{2.11}$$

The main result in the periodic case states that under assumption (2.11) the boundedness of the sequence $(\boldsymbol{A}_*^\varepsilon)$ defined by (2.3) assures that the limit equation of (2.2) is of the same type. More precisely, we have the following result:

Theorem 2 *Let $(\boldsymbol{A}_\sharp^\varepsilon)$ be a sequence of Y-periodic symmetric fourth-order tensor-valued functions satisfying (2.1) for some sequence $(\beta_\varepsilon^\sharp)$ in $\mathrm{L}^1(Y)$ and such that the corresponding sequence $(\boldsymbol{A}_*^\varepsilon)$, given by (2.3), converges to some fourth-order tensor $\boldsymbol{A}^*$. Suppose that $\beta_\varepsilon^\sharp$ is Y-periodic and satisfies*

$$\lim_{\varepsilon \to 0} \left(\varepsilon^2 \int_Y \beta_\varepsilon^\sharp(y) \, dy \right) = 0. \tag{2.12}$$

Then, for any $\boldsymbol{f} \in \mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$, the solution $\boldsymbol{u}^\varepsilon$ of the problem (2.2), with

$$\boldsymbol{A}^\varepsilon(x) := \boldsymbol{A}_\sharp^\varepsilon\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. } x \in \Omega,$$

satisfies

$$\begin{cases} \boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u} & \text{weakly in } \mathrm{H}_0^1(\Omega, \mathbb{R}^2) \\ \boldsymbol{A}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) \rightharpoonup \boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u}) & \text{weakly } * \text{ in } \mathcal{M}(\Omega, \mathbb{R}_s^{2\times 2}), \end{cases} \tag{2.13}$$

where $\boldsymbol{u}$ is the solution in $\mathrm{H}_0^1(\Omega, \mathbb{R}^2)$ of the elasticity problem

$$\begin{cases} -\mathrm{Div}\,(\boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u})) = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Remark 3 A similar result was obtained by Casado-Díaz and the first author in the conduction setting (see [11] for the periodic case and [12] for the non periodic case), under the assumption of the L^1-boundedness of the conductivity coefficients. Theorem 2 is an extension of these results to the elasticity case with the refinement that the bound from above β_ε is not necessarily bounded in $\mathrm{L}^1(Y)$ but satisfies the weaker assumption (2.12). Indeed, the periodicity allows us to relax the L^1-boundedness of the elasticity coefficients. However, we do not know whether condition (2.12) can be improved or even dropped. In any case, it is essential in our proof.

On the other hand, Casado-Díaz and the first author proved in [13] that the result of Theorem 2 holds true in conduction under the only assumption of equicoercivity. More precisely, consider an equicoercive sequence $(A^{\varepsilon}_{\sharp})$ of Y-periodic conductivity matrix-valued functions defined in $\mathbb{R}^2$, and let A^{ε}_{*} be the constant matrix defined by the minimization

$$A^{\varepsilon}_{*}\xi\cdot\xi = \min\left\{\int_Y A^{\varepsilon}_{\sharp}(y)\,(\xi+\nabla\varphi)\cdot(\xi+\nabla\varphi)\;dy \;:\; \varphi\in \mathrm{H}^1_{\#}(Y)\right\},\ \forall\,\xi\in\mathbb{R}^2,$$

i.e., the equivalent of (2.3) for the conduction. In [13] it is proved that the solution u_ε of the conduction problem with conductivity $A^{\varepsilon}_{\sharp}\left(\frac{x}{\varepsilon}\right)$, weakly converges in $\mathrm{H}^1_0(\Omega)$ to the solution u_0 of the conduction problem with a constant conductivity $A^* := \lim A^{\varepsilon}_{*}$. In the case where $A^*\xi\cdot\xi = \infty$ for some direction $\xi\in\mathbb{R}^2\setminus\{(0,0)\}$, the limit equation reduces to $u_0 = 0$ in Ω, due to the boundary condition satisfied by u_ε. This limit potential can be regarded as the solution of a conduction problem with an infinite conductivity in the direction ξ. Therefore, whatever the asymptotic behaviour of A^{ε}_{*}, the limit problem is a conduction problem of the same nature. This corresponds to the notion of compactness defined in the introduction.

It is then natural to ask whether this compactness in two-dimensional conduction still holds true in the elasticity setting. The answer is negative as the following result shows:

Theorem 3 *There exists a sequence* $(\boldsymbol{A}^{\varepsilon})$ *of symmetric fourth-order tensor-valued functions satisfying (2.1) such that the sequence of solutions* $(\boldsymbol{u}^{\varepsilon})$ *of (2.2) strongly converges in* $\mathrm{L}^2(\Omega,\mathbb{R}^2)$ *to a function* $\boldsymbol{u}$ *solving a fourth-order derivatives problem which is thus not an elasticity problem of the type (2.14).*

Remark 4 This result points out a fundamental difference between the conduction case [13] and the elasticity one. Contrary to the conduction case (see the end of Remark 3 above), the sequence of elasticity problems (2.2), with the elasticity tensors $(\boldsymbol{A}^{\varepsilon})$ of Theorem 3, converges in a suitable sense to a limit problem of different nature with the appearance of fourth-order derivatives of the displacement (see Corollary 1 below). We may conclude that generally the compactness result in the two-dimensional conduction case does not extend to the elasticity setting.

The counterexample of Theorem 3 shows that, if we do not assume the boundedness of the sequence (A^*_{ε}), one can loose the compactness of Theorem 2. Therefore, in order to close the periodic framework one has to answer the following question: Does the sole boundedness of (A^*_{ε}) ensure the compactness of Theorem 2? For the moment, we have not succeeded in giving a response to this question.

3 Proofs of the results

This section is devoted to the proofs of the results. We start by proving Theorem 1 using the div-curl lemma 1. Then, we give the proof of Lemma 2 which is the main ingredient of our result in the periodic case. The proof of Theorem 2 follows. The last subsection is devoted to the proof of the counterexample (Theorem 3).

3.1 Proof of Theorem 1

Putting $\boldsymbol{u}^{\varepsilon}$ as a test function in the elasticity problem (2.2) we have, by the α-coercivity of $\boldsymbol{A}^{\varepsilon}$,

$$\alpha\|\boldsymbol{e}(\boldsymbol{u}^{\varepsilon})\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{2\times 2}_s)} \leq \|\boldsymbol{f}\|_{\mathrm{H}^{-1}(\Omega,\mathbb{R}^2)}\|\boldsymbol{u}^{\varepsilon}\|_{\mathrm{L}^2(\Omega,\mathbb{R}^2)}.$$

This, combined with the Poincaré inequality and the Korn inequality, implies that $(\boldsymbol{u}^{\varepsilon})$ is bounded in $\mathrm{H}^1_0(\Omega,\mathbb{R}^2)$. Therefore, the sequence $(\boldsymbol{u}^{\varepsilon})$ weakly converges to some $\boldsymbol{u}$ in $\mathrm{H}^1_0(\Omega,\mathbb{R}^2)$

and $(\nabla \boldsymbol{u}^\varepsilon)$ weakly converges to $\nabla \boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^{2\times 2})$. We then follow the steps of [12] which easily extend to the elasticity case. For the reader's convenience we outline these steps here and refer to [12] for further details.

The first step consists in proving the convergence of the operator $\boldsymbol{\mathcal{A}}^\varepsilon := -\mathrm{Div}\,(\boldsymbol{A}^\varepsilon \nabla\cdot)$ from $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ into $\mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$, which is invertible due to (2.1). Its inverse $\boldsymbol{\mathcal{B}}^\varepsilon$ is bounded by α^{-1}. Hence, the separability of $\mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$ combined with a diagonal extraction process (see [25]) implies that there exist a subsequence, still denoted by ε, and a linear operator $\boldsymbol{\mathcal{B}}$ satisfying $\|\boldsymbol{\mathcal{B}}\| \leq \alpha^{-1}$, such that

$$\forall\, f \in \mathrm{H}^{-1}(\Omega, \mathbb{R}^2), \quad \boldsymbol{\mathcal{B}}^\varepsilon f \rightharpoonup \boldsymbol{\mathcal{B}} f \quad \text{weakly in } \mathrm{H}^1_0(\Omega, \mathbb{R}^2).$$

Moreover, following [12] one can check that the operator $\boldsymbol{\mathcal{B}}$ is coercive with coercivity constant $(\|\beta\|_{\mathrm{L}^\infty(\Omega)})^{-1}$. Therefore, $\boldsymbol{\mathcal{B}}$ is an invertible operator from $\mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$ onto $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ and $\boldsymbol{\mathcal{A}} := \boldsymbol{\mathcal{B}}^{-1}$ satisfies $\|\boldsymbol{\mathcal{A}}\| \leq \|\beta\|_{\mathrm{L}^\infty(\Omega)}$.

The second step provides the construction of the homogenized fourth-order tensor $\boldsymbol{A}^*$. Consider an open subset $\tilde{\Omega}$ of $\mathbb{R}^2$ such that $\Omega \Subset \tilde{\Omega}$. Define, on $\tilde{\Omega}$ the symmetric fourth-order tensor-valued function $\tilde{\boldsymbol{A}}^\varepsilon$ by

$$\tilde{\boldsymbol{A}}^\varepsilon := \begin{cases} \boldsymbol{A}^\varepsilon & \text{in} \quad \Omega \\ \alpha \boldsymbol{I}_4 & \text{in} \quad \tilde{\Omega} \setminus \Omega, \end{cases}$$

where $\boldsymbol{I}_4$ is the identity fourth-order tensor. Let $\tilde{\boldsymbol{\mathcal{A}}}^\varepsilon$ be the operator defined from $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ to $\mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$ by $\tilde{\boldsymbol{\mathcal{A}}}^\varepsilon := -\mathrm{Div}(\tilde{\boldsymbol{A}}^\varepsilon \nabla\cdot)$ and $\tilde{\boldsymbol{\mathcal{B}}}^\varepsilon$ its inverse. From the first step, the sequence $(\tilde{\boldsymbol{\mathcal{B}}}^\varepsilon)$ converges to some operator $\tilde{\boldsymbol{\mathcal{B}}}$ satisfying $\|\tilde{\boldsymbol{\mathcal{B}}}\| \leq \alpha^{-1}$. Additionally, $\tilde{\boldsymbol{\mathcal{A}}} := \tilde{\boldsymbol{\mathcal{B}}}^{-1}$ satisfies $\|\tilde{\boldsymbol{\mathcal{A}}}\| \leq \|\beta\|_{\mathrm{L}^\infty(\Omega)}$. We define, for any $i \in \{1,2,3\}$ the function $\boldsymbol{w}^\varepsilon_{\xi^i} \in \mathrm{H}^1_0(\Omega, \mathbb{R}^2)$, by

$$\boldsymbol{w}^\varepsilon_{\xi^i}(x) := \tilde{\boldsymbol{\mathcal{B}}}^\varepsilon \circ \tilde{\boldsymbol{\mathcal{A}}} \left(\rho(x) \boldsymbol{\xi}^i x\right),$$

where $(\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \boldsymbol{\xi}^3)$ is the canonical basis of $\mathbb{R}^{2\times 2}_s$, and $\rho \in \mathcal{D}(\tilde{\Omega})$ is a cut-off function such that $\rho = 1$ in Ω. Since

$$(\boldsymbol{A}^\varepsilon)^{-1} \left(\boldsymbol{A}^\varepsilon \nabla \boldsymbol{w}^\varepsilon_{\xi^i}\right) : \left(\boldsymbol{A}^\varepsilon \nabla \boldsymbol{w}^\varepsilon_{\xi^i}\right) = \boldsymbol{A}^\varepsilon \nabla \boldsymbol{w}^\varepsilon_{\xi^i} : \nabla \boldsymbol{w}^\varepsilon_{\xi^i}$$

is bounded in $\mathrm{L}^1(\Omega)$, Lemma 1 implies that there exists a subsequence of ε, still denoted by ε, such that

$$\boldsymbol{A}^\varepsilon \nabla \boldsymbol{w}^\varepsilon_{\xi^i} \rightharpoonup \tilde{\boldsymbol{\sigma}}^i \qquad \text{weakly } * \text{ in } \mathcal{M}(\Omega, \mathbb{R}^{2\times 2}_s) \quad \text{for } i \in \{1,2,3\},$$

where $\tilde{\boldsymbol{\sigma}}^i \in \mathrm{L}^2(\Omega, \mathbb{R}^{2\times 2}_s)$. This allows us to define the tensor $\boldsymbol{A}^*$ by

$$\boldsymbol{A}^* \boldsymbol{\xi}^i := \tilde{\boldsymbol{\sigma}}^i \qquad \text{for } i \in \{1,2,3\}.$$

The third step establishes the link between the operator $\boldsymbol{\mathcal{A}}$ and the tensor $\boldsymbol{A}^*$ thanks to Lemma 1. Define $\boldsymbol{\sigma}^\varepsilon := \boldsymbol{A}^\varepsilon \nabla \boldsymbol{u}^\varepsilon$ and $\boldsymbol{v}^\varepsilon := \boldsymbol{w}^\varepsilon_{\xi^i}$. By Lemma 1 there exists $\boldsymbol{\sigma} \in \mathrm{L}^2(\Omega, \mathbb{R}^{2\times 2}_s)$ such that $(\boldsymbol{\sigma}^\varepsilon)$ weakly $*$ converges to $\boldsymbol{\sigma}$ in $\mathcal{M}(\Omega, \mathbb{R}^{2\times 2}_s)$ and

$$\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{\sigma} : \boldsymbol{\xi}^i \qquad \text{in } \mathcal{D}'(\Omega)$$

since by definition $(\boldsymbol{v}^\varepsilon)$ weakly converges to $\boldsymbol{\xi}^i$ in $\mathrm{H}^1(\Omega, \mathbb{R}^2)$. Similarly, we get

$$\boldsymbol{A}^\varepsilon \nabla \boldsymbol{w}^\varepsilon_{\xi^i} : \nabla \boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{A}^* \boldsymbol{\xi}^i : \nabla \boldsymbol{u} = \boldsymbol{A}^* \nabla \boldsymbol{u} : \boldsymbol{\xi}^i \qquad \text{in } \mathcal{D}'(\Omega).$$

From the previous two convergences we deduce that $\boldsymbol{\sigma} = \boldsymbol{A}^* \nabla \boldsymbol{u} = \boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u})$. It comes that $\boldsymbol{\mathcal{A}}(\boldsymbol{v}) = -\mathrm{Div}(\boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{v}))$ for any $\boldsymbol{v} \in \mathrm{H}^1_0(\Omega, \mathbb{R}^2)$. The coercivity and the boundedness of $\boldsymbol{\mathcal{A}}$ imply that $\boldsymbol{A}^*$ satisfies (2.5). This completes the proof of Theorem 1. $\square$

3.2 Proof of Lemma 2

Let $\varphi \in \mathcal{D}(\omega)$ and let $Q_\varepsilon \subset \omega$ be a finite union of squares of the type $\varepsilon(k+\overline{Y})$ containing the support of φ. Define $K_\varepsilon := \{k \in \mathbb{Z}^2,\ \varepsilon(k+\overline{Y}) \subset Q_\varepsilon\}$ and

$$\bar{v}^\varepsilon(x) := \sum_{k\in K_\varepsilon} \left(\fint_{\varepsilon(k+Y)} v^\varepsilon \, dx \right) \mathbf{1}_{\varepsilon(k+Y)}(x), \tag{3.1}$$

where $\mathbf{1}_{\varepsilon(k+Y)}$ is the characteristic function of the set $\varepsilon(k+Y)$. The sequel of the proof is divided in three steps.

First step: Estimate of $\|v^\varepsilon - \bar{v}^\varepsilon\|_{\mathrm{L}^2(Q_\varepsilon)}$.
By the Sobolev inequality corresponding to the continuous embedding of $W^{1,1}(Y)$ into $L^2(Y)$, combined with the Poincaré-Wirtinger inequality in $W^{1,1}(Y)$, there exists a positive constant C such that

$$\int_Y \left(V - \fint_Y V \right)^2 dy \le C \left(\int_Y |\nabla V| \, dy \right)^2 \qquad \forall V \in W^{1,1}(Y).$$

Now, using the change of variables $x = \varepsilon(y+k)$ and $v(x) = V(y)$, for any $\varepsilon > 0$ and $k \in \mathbb{Z}^2$, we have, with the same constant C,

$$\int_{\varepsilon(k+Y)} \left(v - \fint_{\varepsilon(k+Y)} v \right)^2 dx \le C \left(\int_{\varepsilon(k+Y)} |\nabla v| \, dx \right)^2 \qquad \forall v \in W^{1,1}\big(\varepsilon(k+Y)\big).$$

Therefore, by the Cauchy-Schwarz inequality and the periodicity of $\beta_\varepsilon^\sharp$, we have for any $k \in K_\varepsilon$,

$$\begin{aligned} \int_{\varepsilon(k+Y)} (v^\varepsilon - \bar{v}^\varepsilon)^2 \, dx &\le C \left(\int_{\varepsilon(k+Y)} |\nabla v^\varepsilon| \, dx \right)^2 \\ &\le C \left(\int_{\varepsilon(k+Y)} \beta_\varepsilon^\sharp \left(\frac{x}{\varepsilon}\right) dx \right) \left(\int_{\varepsilon(k+Y)} (\beta_\varepsilon^\sharp)^{-1} \left(\frac{x}{\varepsilon}\right) |\nabla v^\varepsilon|^2 \, dx \right) \\ &= C \left(\varepsilon^2 \int_Y \beta_\varepsilon^\sharp \, dy \right) \left(\int_{\varepsilon(k+Y)} (\beta_\varepsilon^\sharp)^{-1} \left(\frac{x}{\varepsilon}\right) |\nabla v^\varepsilon|^2 \, dx \right). \end{aligned} \tag{3.2}$$

Summing the estimates (3.2) over $k \in K_\varepsilon$, and taking into account assumptions (2.10) and (2.12), we obtain

$$\|v^\varepsilon - \bar{v}^\varepsilon\|_{\mathrm{L}^2(Q_\varepsilon)} = o(1). \tag{3.3}$$

Second step: Approximation of a regular function by piecewise-constant ones.
Let $(\bar{\varphi}^\varepsilon)$ be the sequence of piecewise-constant functions defined by

$$\bar{\varphi}^\varepsilon(x) := \sum_{k\in K_\varepsilon} \left(\fint_{\varepsilon(k+Y)} \varphi \, dy \right) \mathbf{1}_{\varepsilon(k+Y)}(x).$$

We claim that

$$\lim_{\varepsilon\to 0} \int_{Q_\varepsilon} \bar{v}^\varepsilon(x) V^\varepsilon \left(\frac{x}{\varepsilon}\right) (\varphi - \bar{\varphi}^\varepsilon)(x) \, dx = 0. \tag{3.4}$$

Indeed, since $\|\varphi - \bar{\varphi}^\varepsilon\|_\infty = O(\varepsilon)$, from the definition (3.1) of the piecewise-constant function $\bar{v}^\varepsilon$ and the Y-periodicity of V^ε, we obtain the estimate

$$
\begin{aligned}
\left|\int_{Q_\varepsilon} \bar{v}^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)(\varphi-\bar{\varphi}^\varepsilon)(x)\,dx\right| &\le O(\varepsilon)\int_{Q_\varepsilon} |\bar{v}^\varepsilon(x)|\left|V^\varepsilon\left(\frac{x}{\varepsilon}\right)\right|\,dx \\
&= O(\varepsilon)\sum_{k\in K_\varepsilon}\int_{\varepsilon(k+Y)}\left|\fint_{\varepsilon(k+Y)} v^\varepsilon(x)\,dx\right|\left|V^\varepsilon\left(\frac{x}{\varepsilon}\right)\right|\,dx \\
&= O(\varepsilon)\left(\int_Y |V^\varepsilon(y)|\,dy\right)\sum_{k\in K_\varepsilon}\left|\int_{\varepsilon(k+Y)} v^\varepsilon(x)\,dx\right| \\
&\le O(\varepsilon)\sum_{k\in K_\varepsilon}\int_{\varepsilon(k+Y)} |v^\varepsilon(x)|\,dx \\
&\le O(\varepsilon)\int_{Q_\varepsilon} |v^\varepsilon(x)|\,dx \le O(\varepsilon)\int_\omega |v^\varepsilon(x)|\,dx.
\end{aligned}
$$

Then, by the Poincaré-Wirtinger inequality in $\mathrm{W}^{1,1}(\omega)$, the Cauchy-Schwarz inequality and assumptions (2.9), (2.10) and (2.12), we obtain the estimate

$$
\begin{aligned}
\left|\int_{Q_\varepsilon} \bar{v}^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)(\varphi-\bar{\varphi}^\varepsilon)(x)\,dx\right| &\le O(\varepsilon)\int_\omega |v^\varepsilon(x)|\,dx \le O(\varepsilon)\int_\omega |\nabla v^\varepsilon(x)|\,dx \\
&\le O(\varepsilon)\left(\int_\omega \beta_\varepsilon^\sharp\left(\frac{x}{\varepsilon}\right)\,dx\right)^{\frac{1}{2}}\left(\int_\omega (\beta_\varepsilon^\sharp)^{-1}\left(\frac{x}{\varepsilon}\right)|\nabla v^\varepsilon(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\le O(1)\left(\varepsilon^2\int_Y \beta_\varepsilon^\sharp(y)\,dy\right)^{\frac{1}{2}}\left(\int_\omega (\beta_\varepsilon^\sharp)^{-1}\left(\frac{x}{\varepsilon}\right)|\nabla v^\varepsilon(x)|^2\,dx\right)^{\frac{1}{2}}
\end{aligned}
$$

which tends to zero by (2.10) and (2.12). The claim (3.4) is proved.

Third step: Conclusion.
Since the support of φ is included in Q_ε, we have

$$
\begin{aligned}
\int_\omega v^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx &= \underbrace{\int_{Q_\varepsilon} (v^\varepsilon(x)-\bar{v}^\varepsilon(x)) V^\varepsilon\left(\frac{x}{\varepsilon}\right)\varphi(x)\,dx}_{I_1^\varepsilon} \\
&\quad + \underbrace{\int_{Q_\varepsilon} \bar{v}^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)(\varphi(x)-\bar{\varphi}^\varepsilon(x))\,dx}_{I_2^\varepsilon} + \underbrace{\int_{Q_\varepsilon} \bar{v}^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)\bar{\varphi}^\varepsilon(x)\,dx}_{I_3^\varepsilon}.
\end{aligned}
$$

Due to (3.3) and (3.4) we have

$$
\lim_{\varepsilon\to 0} I_1^\varepsilon = \lim_{\varepsilon\to 0} I_2^\varepsilon = 0. \tag{3.5}
$$

Moreover, I_3^ε can be rewritten as

$$
\begin{aligned}
I_3^\varepsilon &= \sum_{k\in K_\varepsilon}\int_{\varepsilon(k+Y)} \bar{v}^\varepsilon(x) V^\varepsilon\left(\frac{x}{\varepsilon}\right)\bar{\varphi}^\varepsilon(x)\,dx \\
&= \sum_{k\in K_\varepsilon}\left(\bar{v}^\varepsilon_{|\varepsilon(k+Y)}\right)\left(\bar{\varphi}^\varepsilon_{|\varepsilon(k+Y)}\right)\int_{\varepsilon(k+Y)} V^\varepsilon\left(\frac{x}{\varepsilon}\right)\,dx.
\end{aligned}
$$

Assumption (2.9) combined with the Y-periodicity of V^ε implies that $I_3^\varepsilon = 0$. This, together with (3.5), concludes the proof of Lemma 2.

□

With Lemma 2 proved, we are now in a position to provide the proof of our main homogenization result in the periodic case (Theorem 2).

3.3 Proof of Theorem 2

As in the proof of Theorem 1 the sequence $(\boldsymbol{u}^\varepsilon)$ weakly converges to some $\boldsymbol{u}$ in $\mathrm{H}_0^1(\Omega, \mathbb{R}^2)$ and $(\nabla \boldsymbol{u}^\varepsilon)$ weakly converges to $\nabla \boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^{2\times 2})$. Let $\boldsymbol{\xi}$ be a fixed element in $\mathbb{R}_s^{2\times 2}$. Consider $\boldsymbol{W}_\xi^\varepsilon$ to be the solution of the elasticity problem

$$\left\{\begin{array}{rcll} \mathrm{Div}\left(\boldsymbol{A}_\sharp^\varepsilon\, \boldsymbol{e}(\boldsymbol{W}_\xi^\varepsilon)\right) & = & \boldsymbol{0} & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ y \mapsto \boldsymbol{W}_\xi^\varepsilon(y) - \boldsymbol{\xi} y & & & \text{is } Y\text{-periodic with zero } Y\text{-average.} \end{array}\right. \tag{3.6}$$

For a.e. $x \in \Omega$ and $y \in \mathbb{R}^2$, we set

$$\begin{gathered} \boldsymbol{\sigma}^\varepsilon(x) := \boldsymbol{A}_\sharp^\varepsilon\left(\frac{x}{\varepsilon}\right) \boldsymbol{e}(\boldsymbol{u}^\varepsilon)(x), \quad \boldsymbol{\Sigma}_\xi^\varepsilon(y) := \boldsymbol{A}_\sharp^\varepsilon(y) \boldsymbol{e}(\boldsymbol{W}_\xi^\varepsilon)(y), \quad \boldsymbol{\sigma}_\xi^\varepsilon(x) := \boldsymbol{\Sigma}_\xi^\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{and} \\ \boldsymbol{w}_\xi^\varepsilon(x) := \varepsilon \boldsymbol{W}_\xi^\varepsilon\left(\frac{x}{\varepsilon}\right). \end{gathered} \tag{3.7}$$

Note that $\boldsymbol{\sigma}^\varepsilon(x) := \boldsymbol{A}_\sharp^\varepsilon\left(\frac{x}{\varepsilon}\right) \nabla \boldsymbol{u}^\varepsilon(x)$ and $\boldsymbol{\sigma}_\xi^\varepsilon(x) := \boldsymbol{A}_\sharp^\varepsilon\left(\frac{x}{\varepsilon}\right) \nabla \boldsymbol{w}_\xi^\varepsilon(x)$. By the α-coercivity of $\boldsymbol{A}_\sharp^\varepsilon$ and the boundedness of $\boldsymbol{A}_*^\varepsilon$, we have

$$\alpha \|\boldsymbol{e}(\boldsymbol{W}_\xi^\varepsilon)\|^2_{\mathrm{L}^2(Y, \mathbb{R}_s^{2\times 2})} \leq \int_Y \boldsymbol{A}_\sharp^\varepsilon(y) \boldsymbol{e}(\boldsymbol{W}_\xi^\varepsilon) : \boldsymbol{e}(\boldsymbol{W}_\xi^\varepsilon)\, dy = \boldsymbol{A}_*^\varepsilon \boldsymbol{\xi} : \boldsymbol{\xi} \leq c. \tag{3.8}$$

This, combined with the Poincaré-Wirtinger inequality and the Korn inequality, implies that $\boldsymbol{W}_\xi^\varepsilon - \boldsymbol{\xi} y$ and thus $\boldsymbol{W}_\xi^\varepsilon$ are bounded in $\mathrm{H}_\#^1(Y, \mathbb{R}^2)$. Therefore, we get

$$\boldsymbol{w}_\xi^\varepsilon(x) \rightharpoonup \boldsymbol{\xi} x \quad \text{weakly in } \mathrm{H}^1(\Omega, \mathbb{R}^2). \tag{3.9}$$

Taking into account the div-curl Lemma 2, we apply the method of oscillating test functions due to Tartar [28], which consists in determining the limit of the energy $\boldsymbol{A}_\sharp^\varepsilon(\frac{x}{\varepsilon}) \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{w}_\xi^\varepsilon$ in the sense of distributions. The sequel of the proof is divided in three steps.

First step: Limit of $\boldsymbol{\sigma}^\varepsilon : \boldsymbol{e}(\boldsymbol{w}_\xi^\varepsilon) = \boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}_\xi^\varepsilon$
Let $\varphi \in \mathcal{D}(\Omega)$. Using a localization procedure, we can assume that $\mathrm{supp}(\varphi)$ in contained in a regular simply connected domain ω of Ω. Let $\boldsymbol{v} \in \mathrm{H}_0^1(\Omega, \mathbb{R}^2)$ be the solution of the problem $\Delta \boldsymbol{v} = \boldsymbol{f}$ in $\mathcal{D}'(\Omega, \mathbb{R}^2)$. Since $(\boldsymbol{\sigma}^\varepsilon - \nabla \boldsymbol{v})$ is divergence free, there exists (see, for instance, [20]) a sequence of stream functions $(\boldsymbol{v}^\varepsilon)$ in $\mathrm{H}^1(\omega, \mathbb{R}^2)$, satisfying

$$\boldsymbol{\sigma}^\varepsilon = \nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} \quad \text{a.e. in } \omega \qquad \text{with} \qquad \fint_\omega \boldsymbol{v}^\varepsilon\, dx = \boldsymbol{0}, \tag{3.10}$$

where $\boldsymbol{J}$ is the matrix associated with the rotation by $\pi/2$, *i.e.,*

$$\boldsymbol{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that, although $\boldsymbol{\sigma}^\varepsilon$ in (3.10) is symmetric, it is the whole gradient of the stream function $\boldsymbol{v}^\varepsilon$ and not only its symmetric part $\boldsymbol{e}(\boldsymbol{v}^\varepsilon)$, that appears in (3.10). Let Q_ε and K_ε be chosen as in the proof of Lemma 2. We have

$$\int_\Omega (\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}_\xi^\varepsilon)\, \varphi\, dx = \int_\Omega (\nabla \boldsymbol{v} : \nabla \boldsymbol{w}_\xi^\varepsilon)\, \varphi\, dx + \int_\Omega (\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} : \nabla \boldsymbol{w}_\xi^\varepsilon)\, \varphi\, dx. \tag{3.11}$$

By the convergence (3.9) of $\boldsymbol{w}_\xi^\varepsilon$, the first term of the left-hand side of (3.11) converges to $\int_\Omega (\nabla \boldsymbol{v} : \boldsymbol{\xi})\, \varphi\, dx$, hence

$$\int_\Omega (\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}_\xi^\varepsilon)\, \varphi\, dx = \int_\Omega (\nabla \boldsymbol{v} : \boldsymbol{\xi})\, \varphi\, dx + \int_\Omega (\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} : \nabla \boldsymbol{w}_\xi^\varepsilon)\, \varphi\, dx + o(1). \tag{3.12}$$

By integrating by parts and using the antisymmetry of $\boldsymbol{J}$, we have

$$\begin{aligned}\int_\Omega (\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} : \nabla \boldsymbol{w}^\varepsilon_\xi)\,\varphi\,dx &= -\int_\Omega (\nabla \boldsymbol{v}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi \boldsymbol{J})\,\varphi\,dx \\ &= -\int_\Omega \nabla(\varphi \boldsymbol{v}^\varepsilon) : \nabla \boldsymbol{w}^\varepsilon_\xi \boldsymbol{J}\,dx + \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \nabla\varphi) : \nabla \boldsymbol{w}^\varepsilon_\xi \boldsymbol{J}\,dx. \end{aligned} \tag{3.13}$$

On the one hand, since $\nabla \boldsymbol{w}^\varepsilon_\xi \boldsymbol{J}$ is divergence free, the first term of the right-hand side of (3.13) is equal to zero. On the other hand, the boundedness of $\boldsymbol{u}^\varepsilon$ in $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ combined with (2.1) yields

$$\begin{aligned}\int_\Omega (\beta^\sharp_\varepsilon)^{-1}\left(\frac{x}{\varepsilon}\right)\|\nabla \boldsymbol{v}^\varepsilon\|^2\,dx &= \int_\Omega (\beta^\sharp_\varepsilon)^{-1}\left(\frac{x}{\varepsilon}\right)\|\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J}\|^2\,dx \\ &\le 2\int_\Omega (\beta^\sharp_\varepsilon)^{-1}\left(\frac{x}{\varepsilon}\right)\|\boldsymbol{\sigma}^\varepsilon\|^2\,dx + 2\int_\Omega (\beta^\sharp_\varepsilon)^{-1}\left(\frac{x}{\varepsilon}\right)\|\nabla \boldsymbol{v}\|^2\,dx \\ &\le 2\int_\Omega \boldsymbol{A}^\varepsilon_\sharp\left(\frac{x}{\varepsilon}\right)\boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \boldsymbol{e}(\boldsymbol{u}^\varepsilon)dx + 2\int_\Omega (\beta^\sharp_\varepsilon)^{-1}\left(\frac{x}{\varepsilon}\right)\|\nabla \boldsymbol{v}\|^2 dx \\ &\le 2c\,\|\boldsymbol{u}^\varepsilon\|_{\mathrm{H}^1_0(\Omega,\mathbb{R}^2)}\|f\|_{\mathrm{H}^{-1}(\Omega,\mathbb{R}^2)} + 2\alpha^{-1}\int_\Omega \|\nabla \boldsymbol{v}\|^2\,dx \le c. \end{aligned} \tag{3.14}$$

Moreover, $\boldsymbol{v}^\varepsilon$ and $(\nabla \boldsymbol{W}^\varepsilon_\xi - \boldsymbol{\xi})$ have zero average-value in ω and Y, respectively. Then, the sequences $\boldsymbol{v}^\varepsilon$ and $(\nabla \boldsymbol{W}^\varepsilon_\xi - \boldsymbol{\xi})$ satisfy the assumptions of Lemma 2. Hence, we get

$$\int_\Omega (\boldsymbol{v}^\varepsilon \otimes \nabla\varphi) : \big(\nabla \boldsymbol{w}^\varepsilon_\xi - \boldsymbol{\xi}\big)\,\boldsymbol{J}\,dx = o(1).$$

Therefore, from (3.13) we can write

$$\int_\Omega (\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} : \nabla \boldsymbol{w}^\varepsilon_\xi)\,\varphi\,dx = \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \nabla\varphi) : \boldsymbol{\xi}\boldsymbol{J}\,dx + o(1). \tag{3.15}$$

Taking into account (3.12) and (3.15), we have

$$\begin{aligned}\int_\Omega (\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi)\,\varphi\,dx &= \int_\Omega (\nabla \boldsymbol{v} : \boldsymbol{\xi})\,\varphi\,dx + \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \nabla\varphi) : \boldsymbol{\xi}\boldsymbol{J}\,dx + o(1) \\ &= \int_\Omega (\nabla \boldsymbol{v} : \boldsymbol{\xi})\,\varphi\,dx + \int_\Omega (\nabla(\varphi \boldsymbol{v}^\varepsilon) - \varphi \nabla \boldsymbol{v}_\varepsilon) : \boldsymbol{\xi}\boldsymbol{J}\,dx + o(1) \\ &= \int_\Omega (\nabla \boldsymbol{v} : \boldsymbol{\xi})\,\varphi\,dx + \int_\Omega (\nabla \boldsymbol{v}^\varepsilon \boldsymbol{J} : \boldsymbol{\xi})\,\varphi\,dx + o(1) \\ &= \int_\Omega (\boldsymbol{\sigma}^\varepsilon : \boldsymbol{\xi})\,\varphi\,dx + o(1). \end{aligned}$$

Hence,

$$\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi - \boldsymbol{\sigma}^\varepsilon : \boldsymbol{\xi} \rightharpoonup 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.16}$$

Moreover, $(\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi)$ is bounded in $\mathrm{L}^1(\Omega)$ since the Cauchy-Schwarz inequality, together with the boundedness (3.8) of $(\boldsymbol{A}^\varepsilon_*)$, implies

$$\begin{aligned}\int_\Omega |\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi|\,dx &= \int_\Omega |\boldsymbol{\sigma}^\varepsilon : \boldsymbol{e}(\boldsymbol{w}^\varepsilon_\xi)|\,dx \\ &\le \left(\int_\Omega \boldsymbol{A}^\varepsilon_\sharp\left(\frac{x}{\varepsilon}\right)\boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \boldsymbol{e}(\boldsymbol{u}^\varepsilon)\,dx\right)^{\frac12}\left(\int_\Omega \boldsymbol{A}^\varepsilon_\sharp\left(\frac{x}{\varepsilon}\right)\boldsymbol{e}(\boldsymbol{w}^\varepsilon_\xi) : \boldsymbol{e}(\boldsymbol{w}^\varepsilon_\xi)\,dx\right)^{\frac12} \\ &\le c\left(\|\boldsymbol{u}^\varepsilon\|_{\mathrm{H}^1_0(\Omega,\mathbb{R}^2)}\|f\|_{\mathrm{H}^{-1}(\Omega,\mathbb{R}^2)}\right)^{\frac12}(\boldsymbol{A}^\varepsilon_*\boldsymbol{\xi} : \boldsymbol{\xi})^{\frac12} \le c, \end{aligned}$$

for some positive constant c. Therefore, up to a subsequence, there exists some $\boldsymbol{\sigma}$ in $\mathcal{M}(\Omega, \mathbb{R}^{2\times 2}_s)$ such that $(\boldsymbol{\sigma}^\varepsilon)$ converges to $\boldsymbol{\sigma}$ in the weak $*$ sense of the Radon measures in Ω, and thus

$$\boldsymbol{\sigma}^\varepsilon \rightharpoonup \boldsymbol{\sigma} \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^{2\times 2}_s). \tag{3.17}$$

Second step: Limit of $\boldsymbol{e}(\boldsymbol{u}^\varepsilon) : \boldsymbol{\sigma}^\varepsilon_\xi = \nabla \boldsymbol{u}^\varepsilon : \boldsymbol{\sigma}^\varepsilon_\xi$

Since $\boldsymbol{\Sigma}^\varepsilon_\xi$ is divergence free, there exists (see, for instance, [20]) a sequence of Y-periodic vector-valued stream functions $(\boldsymbol{V}^\varepsilon_\xi)$ with zero Y-average, such that

$$\boldsymbol{\Sigma}^\varepsilon_\xi = \nabla \boldsymbol{V}^\varepsilon_\xi \boldsymbol{J} + \int_Y \boldsymbol{\Sigma}^\varepsilon_\xi \, dy, \tag{3.18}$$

where $\nabla \boldsymbol{V}^\varepsilon_\xi \boldsymbol{J}$ is divergence free. Note that, from the definitions (2.3) of $\boldsymbol{A}^\varepsilon_*$ and (3.7) of $\boldsymbol{\Sigma}^\varepsilon_\xi$, we have

$$\int_Y \boldsymbol{\Sigma}^\varepsilon_\xi \, dy = \boldsymbol{A}^\varepsilon_* \boldsymbol{\xi}.$$

The boundedness of $(\boldsymbol{A}^\varepsilon_*)$ combined with (2.1) ensures the existence of a positive constant c such that

$$\begin{aligned} \int_Y (\beta^\sharp_\varepsilon)^{-1}(y) \|\nabla \boldsymbol{V}^\varepsilon_\xi\|^2 \, dy &\leq 2 \int_Y (\beta^\sharp_\varepsilon)^{-1}(y) \|\boldsymbol{A}^\varepsilon_* \xi\|^2 \, dy + 2 \int_Y (\beta^\sharp_\varepsilon)^{-1}(y) \|\boldsymbol{\Sigma}^\varepsilon_\xi\|^2 \, dy \\ &\leq c + 2 \int_Y (\boldsymbol{A}^\varepsilon_\sharp)^{-1} \boldsymbol{\Sigma}^\varepsilon_\xi : \boldsymbol{\Sigma}^\varepsilon_\xi \, dy = c + 2 \, \boldsymbol{A}^\varepsilon_* \boldsymbol{\xi} : \boldsymbol{\xi} \leq c. \end{aligned} \tag{3.19}$$

Define the oscillating sequence $\boldsymbol{v}^\varepsilon_\xi(x) := \varepsilon \boldsymbol{V}^\varepsilon_\xi \left(\frac{x}{\varepsilon}\right)$. For any $\varphi \in \mathcal{D}(\Omega)$, we have by (3.18)

$$\int_\Omega (\nabla \boldsymbol{u}^\varepsilon : \boldsymbol{\sigma}^\varepsilon_\xi) \, \varphi \, dx = \int_\Omega (\nabla \boldsymbol{u}^\varepsilon : \boldsymbol{A}^\varepsilon_* \boldsymbol{\xi}) \, \varphi \, dx + \int_\Omega (\nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon_\xi \boldsymbol{J}) \, \varphi \, dx. \tag{3.20}$$

Owing to the boundedness of $(\boldsymbol{A}^\varepsilon_*)$ there exist a subsequence of ε, still denoted by ε, and a symmetric fourth-order tensor $\boldsymbol{A}^*$ such that $(\boldsymbol{A}^\varepsilon_*)$ converges to $\boldsymbol{A}^*$. Therefore, the first term of the left hand side of (3.20) clearly converges to $\int_\Omega (\nabla \boldsymbol{u} : \boldsymbol{A}^* \boldsymbol{\xi}) \, \varphi \, dx$. Moreover, since $\nabla \boldsymbol{u}^\varepsilon \boldsymbol{J}$ is divergence free, an integration by parts yields

$$\int_\Omega (\nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon_\xi \boldsymbol{J}) \, \varphi \, dx = \int_\Omega (\boldsymbol{v}^\varepsilon_\xi \otimes \boldsymbol{J} \nabla \varphi) : \nabla \boldsymbol{u}^\varepsilon \, dx. \tag{3.21}$$

By the Sobolev-Poincaré inequality and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|\boldsymbol{v}^\varepsilon_\xi\|_{\mathrm{L}^2(\Omega, \mathbb{R}^2)} &\leq c \, \varepsilon \|\boldsymbol{V}^\varepsilon_\xi\|_{\mathrm{L}^2(Y, \mathbb{R}^2)} \leq c \, \varepsilon \int_Y \|\nabla \boldsymbol{V}^\varepsilon_\xi\| \, dy \\ &\leq c \left(\varepsilon^2 \int_Y \beta^\sharp_\varepsilon(y) \, dy \right)^{\frac{1}{2}} \left(\int_Y (\beta^\sharp_\varepsilon)^{-1}(y) \|\nabla \boldsymbol{V}^\varepsilon_\xi\|^2 \, dy \right)^{\frac{1}{2}}. \end{aligned} \tag{3.22}$$

From the estimates (3.22), (3.19) and assumption (2.12), we deduce the strong convergence of $(\boldsymbol{v}^\varepsilon_\xi)$ to zero in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$. Therefore, putting together (3.20) and (3.21), we get the convergence

$$\nabla \boldsymbol{u}^\varepsilon : \boldsymbol{\sigma}^\varepsilon_\xi \rightharpoonup \nabla \boldsymbol{u} : \boldsymbol{A}^* \boldsymbol{\xi} \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.23}$$

Third step: Conclusion.

Thanks to the symmetry of $\boldsymbol{A}^\varepsilon_\sharp$ we have $\boldsymbol{\sigma}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\xi = \boldsymbol{\sigma}^\varepsilon_\xi : \nabla \boldsymbol{u}^\varepsilon$. The convergences (3.16), (3.17) and (3.23) of the previous two steps imply the equality $\boldsymbol{\sigma} : \boldsymbol{\xi} = \boldsymbol{A}^* \nabla \boldsymbol{u} : \boldsymbol{\xi}$ for any $\boldsymbol{\xi} \in \mathbb{R}^{2\times 2}_s$. Therefore, $\boldsymbol{\sigma} = \boldsymbol{A}^* \nabla \boldsymbol{u} = \boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u})$ in $\mathcal{D}'(\Omega, \mathbb{R}^{2\times 2}_s)$. Recalling that $-\mathrm{Div}(\boldsymbol{\sigma}^\varepsilon) = \boldsymbol{f}$ in $\mathcal{D}'(\Omega, \mathbb{R}^2)$, we obtain that $-\mathrm{Div}(\boldsymbol{\sigma}) = \boldsymbol{f}$ in $\mathcal{D}'(\Omega, \mathbb{R}^2)$. Hence,

$$-\mathrm{Div}\, (\boldsymbol{A}^* \boldsymbol{e}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^2),$$

which concludes the proof of Theorem 2. □

3.4 Proof of Theorem 3

Theorem 3 is a straightforward consequence of Theorem 4 and Corollary 1 below. These two results are based on the homogenization of a two-phase material the phases of which are homogeneous with one being very stiff. We prove that the effective behaviour of this material dramatically differs from the one of its two components. Our construction is inspired by the work of Pideri and Seppecher [26] in dimension three, which is based on a homogeneous material reinforced by a periodic lattice of very stiff and very thin cylindrical fibers. Here, the fibers are replaced by very stiff strips.

3.4.1 Description of the composite material

For the seek of simplicity we assume Ω is the unit square $(0,1)^2$ of $\mathbb{R}^2$. We denote by I the interval $(-1,1)$. Let ε and r_ε be two real numbers such that

$$\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon}=0. \tag{3.24}$$

Let $p_\varepsilon:\mathbb{R}^2\to\mathbb{R}$ be the map defined by $p_\varepsilon(y):=\varepsilon(\mathrm{Int}(y_1/\varepsilon)+1/2)$, for $y=(y_1,y_2)\in\mathbb{R}^2$, where $\mathrm{Int}(\cdot)$ is the integer part function. The map p_ε transforms any strip of the type $\big[j\varepsilon,(j+1)\varepsilon\big)\times\mathbb{R}$, $j\in\mathbb{Z}$, into the point $(j+1/2)\varepsilon$. We then define $y_\varepsilon:\mathbb{R}^2\to\mathbb{R}$ by

$$y_\varepsilon(x_1,x_2):=\frac{x_1-p_\varepsilon(x)}{r_\varepsilon}. \tag{3.25}$$

The reinforced part of the material is defined to be

$$F_\varepsilon:=\{x=(x_1,x_2)\in\Omega:\ |x_1-p_\varepsilon(x)|<r_\varepsilon\},$$

and the remaining part of the domain is denoted by $M_\varepsilon:=\Omega\setminus F_\varepsilon$. Note that

$$p_\varepsilon(\Omega)=\{x_\varepsilon^i:=(i-1/2)\varepsilon:\ 1\le i\le\varepsilon^{-1}\}.$$

By P_ε^i we denote the ith period of the composite material

$$P_\varepsilon^i:=\{x\in\Omega,\ p_\varepsilon(x)=x_\varepsilon^i\}=[x_\varepsilon^i-\varepsilon/2,x_\varepsilon^i+\varepsilon/2)\times(0,1).$$

The strip $F_\varepsilon\cap P_\varepsilon^i$ which is contained in the period P_ε^i is denoted by F_ε^i. Then, the reinforcing zone area is $|F_\varepsilon|=(2r_\varepsilon)/\varepsilon$.

Assume that F_ε and M_ε are occupied by two isotropic elastic materials the Lamé coefficients of which are $(\lambda_\varepsilon,\mu_\varepsilon)$ and (λ,μ), respectively. We assume that λ_ε and μ_ε are of the same order of magnitude, *i.e.*, there exists a nonnegative real number ℓ such that

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{\mu_\varepsilon}=\ell. \tag{3.26}$$

We also assume that the behaviours of the different parameters ε, r_ε, λ_ε, μ_ε are linked by the existence of a positive real number μ_1 such that

$$\lim_{\varepsilon\to 0}\frac{\mu_\varepsilon r_\varepsilon^3}{\varepsilon}=\mu_1. \tag{3.27}$$

The physical meaning of assumption (3.27) is that the bending stiffness of one strip F_ε^i is of the order of ε. Indeed, the bending stiffness of a single strip is (see *e.g.* [29], p. 5)

$$s_\varepsilon=\frac{2E_\varepsilon r_\varepsilon^3}{3(1-\nu_\varepsilon^2)},$$

where ν_ε and E_ε stand for the Poisson coefficient and the Young's modulus of the strip F^i_ε, respectively. Using the fact that

$$E_\varepsilon := 2\mu_\varepsilon(1+\nu_\varepsilon) \quad \text{and} \quad \nu_\varepsilon := \frac{\lambda_\varepsilon}{2(\lambda_\varepsilon+\mu_\varepsilon)},$$

we obtain that the "total bending stiffness" of the composite is

$$k_\varepsilon := \frac{8}{3}\frac{\mu_\varepsilon r_\varepsilon^3}{\varepsilon}\left(\frac{\lambda_\varepsilon+\mu_\varepsilon}{\lambda_\varepsilon+2\mu_\varepsilon}\right).$$

Therefore, to obtain the convergence of this total bending stiffness, as ε goes to zero, we need assumption (3.27). Note that in the three-dimensional case [26] the bending stiffness involves a different power of the radius r_ε, namely r_ε^4.

For any $\boldsymbol{u} \in \mathrm{L}^2(\Omega,\mathbb{R}^2)$ and any Borel set $\omega \subset \Omega$ we define the matrix energy to be

$$E^m(\omega,\boldsymbol{u}) := \int_\omega \left(2\mu\|\boldsymbol{e}(\boldsymbol{u})\|^2 + \lambda\left(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}))\right)^2\right)\, dx.$$

We similarly define the strip energy

$$E^s_\varepsilon(\omega,\boldsymbol{u}) := \int_\omega \left(2\mu_\varepsilon\|\boldsymbol{e}(\boldsymbol{u})\|^2 + \lambda_\varepsilon\left(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}))\right)^2\right)\, dx.$$

Perfect adhesion is assumed between the different components of the composite. We also assume that the displacement on the boundary $\partial\Omega$ of the composite is zero. Finally, for any $\boldsymbol{u} \in \mathrm{L}^2(\Omega,\mathbb{R}^2)$ we define the total energy of the composite to be

$$E_\varepsilon(\Omega,\boldsymbol{u}) := \begin{cases} E^m(M_\varepsilon,\boldsymbol{u}) + E^s_\varepsilon(F_\varepsilon,\boldsymbol{u}) & \text{if } \boldsymbol{u} \in \mathrm{H}^1_0(\Omega,\mathbb{R}^2), \\ \infty & \text{otherwise.} \end{cases}$$

We have the following result:

Theorem 4 *The sequence of energies (E_ε) Γ-converges, in the strong topology of $\mathrm{L}^2(\Omega,\mathbb{R}^2)$, to the limit energy E_0 defined by*

$$E_0(\boldsymbol{u}) := \begin{cases} E^m(\Omega,\boldsymbol{u}) + k\displaystyle\int_\Omega \left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2 dx & \text{if } \boldsymbol{u} \in \mathcal{V}, \\ \infty & \text{otherwise,} \end{cases} \tag{3.28}$$

where

$$\mathcal{V} := \left\{\boldsymbol{u} \in \mathrm{H}^1_0(\Omega,\mathbb{R}^2) : \frac{\partial^2 u_1}{\partial x_2^2} \in \mathrm{L}^2(\Omega), u_2 = 0 \text{ a.e. in } \Omega,\ \frac{\partial u_1}{\partial x_2} = 0 \text{ a.e. on } (0,1)\times\{0,1\}\right\} \tag{3.29}$$

and

$$k := \lim_{\varepsilon\to 0} k_\varepsilon = \frac{8}{3}\mu_1\left(\frac{\ell+1}{\ell+2}\right). \tag{3.30}$$

More precisely,

i) For any sequence $(\boldsymbol{u}^\varepsilon)$ strongly converging to some $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega,\mathbb{R}^2)$, we have the lower-bound inequality

$$\liminf_{\varepsilon\to 0} E_\varepsilon(\Omega,\boldsymbol{u}^\varepsilon) \geq E_0(\boldsymbol{u}).$$

ii) For any $\boldsymbol{u} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$, there exists an approximating sequence $(\boldsymbol{u}^\varepsilon)$ strongly converging to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$, such that we have the upper-bound inequality

$$\limsup_{\varepsilon \to 0} E_\varepsilon(\Omega, \boldsymbol{u}^\varepsilon) \leq E_0(\boldsymbol{u}).$$

Passing from the energy viewpoint of Theorem 4 to the associated Euler equation, it follows:

Corollary 1 *Let $\boldsymbol{A}^\varepsilon$ be the εY-periodic tensor-valued function defined by*

$$\boldsymbol{A}^\varepsilon(x) := 2\big(\mu_\varepsilon \mathbf{1}_{F_\varepsilon}(x) + \mu \mathbf{1}_{M_\varepsilon}(x)\big)\boldsymbol{I}_4 + \big(\lambda_\varepsilon \mathbf{1}_{F_\varepsilon}(x) + \lambda \mathbf{1}_{M_\varepsilon}(x)\big)\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 \quad a.e.\ x \in \Omega. \tag{3.31}$$

Then, the solution $\boldsymbol{u}^\varepsilon$ of the elasticity problem (2.2), with (3.31) and a right-hand side $\boldsymbol{f}$ in $\mathrm{H}^{-1}(\Omega, \mathbb{R}^2)$, weakly converges in $\mathrm{H}^1(\Omega, \mathbb{R}^2)$ to the function $\boldsymbol{u}^0 = (u_1^0, 0) \in \mathcal{V}$, which is the unique solution of the fourth-order equation

$$\begin{cases} -(\lambda+\mu)\dfrac{\partial^2 u_1^0}{\partial x_1^2} - \mu\Delta u_1^0 + k\dfrac{\partial^4 u_1^0}{\partial x_2^4} &= f_1 \quad in\ \Omega \\ u_1^0 &= 0 \quad on\ \ \partial\Omega \\ \dfrac{\partial u_1^0}{\partial x_2} &= 0 \quad on\ (0,1)\times\{0,1\}. \end{cases} \tag{3.32}$$

Remark 5 Contrary to Theorem 2, the constant fourth-order tensor $\boldsymbol{A}_*^\varepsilon$ of (2.3), defined with the Y-periodic tensor-valued function $\boldsymbol{A}_\sharp^\varepsilon(y) := \boldsymbol{A}^\varepsilon(\varepsilon y)$ of (3.31), is not bounded. Indeed, it is easy to check that $\boldsymbol{A}_*^\varepsilon(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) : (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2)$ tends to infinity, where $\boldsymbol{e}_2 := (0,1)$. This unboundedness induces the degenerate limit equation (3.32). This is not at all the case in conduction as shown in [13]. More precisely, in the conduction framework the limit equation is $u_0 = 0$ if the homogenized conductivity matrix A_*^ε is not bounded (see the end of Remark 3 above). In Corollary 1 we also have $u_2^0 = 0$ but the first component u_1^0 is not zero since it satisfies the degenerate equation (3.32).

Proof of Corollary 1. Using a density argument, we are led to the case $\boldsymbol{f} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$. Then, we apply the Γ-convergence of Theorem 4 to the sequence of energies

$$F_\varepsilon(\boldsymbol{u}) := E_\varepsilon(\Omega, \boldsymbol{u}) - 2\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx$$

which Γ-converges to

$$F_0(\boldsymbol{u}) := E_0(\Omega, \boldsymbol{u}) - 2\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx.$$

Moreover, in virtue of the Lax-Milgram Theorem applied in the space $\mathcal{V}$ endowed with the norm

$$\|\boldsymbol{u}\|_{\mathcal{V}} := \|\boldsymbol{u}\|_{\mathrm{H}^1(\Omega,\mathbb{R}^2)} + \left\|\frac{\partial^2 u_1}{\partial x_2^2}\right\|_{\mathrm{L}^2(\Omega)}, \tag{3.33}$$

the functional F_0 has a unique minimum $u^0 = (u_1^0, 0) \in \mathcal{V}$ solution of the equation (3.32). We conclude by passing to the Euler equations since the Γ-convergence implies the convergence of the minimizers due to the equicoercivity of F_ε combined with the uniqueness of the minimum of F_0 (see *e.g.* Theorem 7.8 and Corollary 7.24 of [17]).

□

3.4.2 Proof of Theorem 4

Before proceeding with the proof of Theorem 4, let us give some preliminary results we use in the sequel. These results (Lemma 3 an Lemma 4) are an easy two-dimensional adaptation of those obtained in dimension three by Pideri and Seppecher [26]. Following [26] we define a double-scale convergence well adapted to the geometry of the problem.

Definition 1 *A sequence (u_ε) in $\mathrm{L}^2(\Omega)$ is said to double-scale converge to $v \in \mathrm{L}^2(\Omega \times I)$ if, for any $\varphi \in C_0^\infty(\Omega \times I)$ we have*

$$\lim_{\varepsilon \to 0} \fint_{F_\varepsilon} u_\varepsilon(x)\, \varphi(x, y_\varepsilon(x))\, dx = \fint_\Omega \fint_I v(x,y)\, \varphi(x,y)\, dy dx.$$

A sequence of vectors or tensors will be said to double-scale converge if and only if its components double-scale converge in the sense of Definition 1.

Here we give some double-scale convergence properties for sequences with bounded energy.

Lemma 3 *For any function Ψ in $C_0^\infty(\Omega \times I)$ the sequence $\Psi(\cdot, y_\varepsilon(\cdot))$ double-scale converges to Ψ.*

Lemma 3 states that the constant sequence 1 defined on Ω double-scale converges to the constant function 1 defined on $\Omega \times I$.

Lemma 4 *Assume that (3.24), (3.26) and (3.27) hold true. Let $(\boldsymbol{u}^\varepsilon)$ be a sequence in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$ with bounded energy $E_\varepsilon(\Omega, \boldsymbol{u}^\varepsilon)$.*

i) Then, there exist $\boldsymbol{v} \in \mathrm{L}^2(\Omega \times I, \mathbb{R}^2)$, $w \in \mathrm{L}^2(\Omega \times I)$ and $\boldsymbol{\chi} \in \mathrm{L}^2(\Omega \times I, \mathbb{R}_s^{2\times 2})$ such that, up to a subsequence, the sequences $(\boldsymbol{u}^\varepsilon)$, $(u_2^\varepsilon / r_\varepsilon)$ and $(e(\boldsymbol{u}^\varepsilon)/r_\varepsilon)$ double-scale converge to $\boldsymbol{v}$, w and $\boldsymbol{\chi}$, respectively.

ii) Suppose, in addition, that the sequence $(\boldsymbol{u}^\varepsilon)$ strongly converges to some $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$, then

$$\boldsymbol{u} \in \mathrm{H}_0^1(\Omega, \mathbb{R}^2), \qquad \frac{\partial^2 u_1}{\partial x_2^2} \in \mathrm{L}^2(\Omega) \qquad \text{and} \qquad u_2 = 0 \quad a.e.\ in\ \Omega.$$

Moreover, there exists a function $q \in \mathrm{L}^2(\Omega)$ such that, up to a subsequence, the sequence $(e(\boldsymbol{u}^\varepsilon)/r_\varepsilon)_{22}$ double-scale converges to $q(x) - \frac{\partial^2 u_1}{\partial x_2^2}(x) y$.

For detailed proofs of Lemma 3 and Lemma 4 we refer to [26] where the techniques can be adapted to the two-dimensional setting at the expense of a few changes of order of magnitude. For the reader's convenience we give the main steps of the proof of *ii)*.

Proof of *ii)* of Lemma 4. From the boundedness and the equicoercivity of the energy $(E_\varepsilon(\Omega, \boldsymbol{u}^\varepsilon))$ combined with the Korn inequality, we deduce that, up to a subsequence, $(\boldsymbol{u}^\varepsilon)$ weakly converges to $\boldsymbol{u}$ in $\mathrm{H}_0^1(\Omega, \mathbb{R}^2)$. Moreover, by *i)* there exist $\boldsymbol{v} \in \mathrm{L}^2(\Omega \times I, \mathbb{R}^2)$, $w \in \mathrm{L}^2(\Omega \times I)$ and $\boldsymbol{\chi} \in \mathrm{L}^2(\Omega \times I, \mathbb{R}_s^{2\times 2})$ such that, up to a subsequence, the sequences $(\boldsymbol{u}^\varepsilon)$, $(u_2^\varepsilon / r_\varepsilon)$ and $(e(\boldsymbol{u}^\varepsilon)/r_\varepsilon)$ double-scale converge to $\boldsymbol{v}$, w and $\boldsymbol{\chi}$, respectively. Since r_ε tends to zero, so does u_2^ε, hence

$$v_2 = 0 \qquad \text{a.e. in } \Omega. \tag{3.34}$$

From the strong convergence of $(\boldsymbol{u}^\varepsilon)$ to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$ and the double-scale convergence of $(\boldsymbol{u}^\varepsilon)$ to $\boldsymbol{v}$, we easily get

$$\boldsymbol{u}(x) = \fint_I \boldsymbol{v}(x,y)\, dy \qquad \text{a.e. } x \in \Omega. \tag{3.35}$$

It follows, from (3.34) and (3.35) that

$$u_2 = 0 \qquad \text{a.e. in } \Omega.$$

Now, let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times I, \mathbb{R}^{2\times 2}_s)$. An integration by parts yields

$$\begin{aligned}\frac{1}{r_\varepsilon}\int_{F_\varepsilon} \boldsymbol{e}(\boldsymbol{u}^\varepsilon)(x) : \boldsymbol{\varphi}(x, y_\varepsilon(x))\, dx &= -\frac{1}{r_\varepsilon^2}\int_{F_\varepsilon} u_1^\varepsilon \frac{\partial \varphi_{11}}{\partial y}(x, y_\varepsilon(x))\, dx \\ -\frac{1}{r_\varepsilon}\int_{F_\varepsilon} u_i^\varepsilon \frac{\partial \varphi_{ij}}{\partial x_j}(x, y_\varepsilon(x))\, dx &- \frac{1}{r_\varepsilon^2}\int_{F_\varepsilon} u_2^\varepsilon \frac{\partial \varphi_{21}}{\partial y}(x, y_\varepsilon(x))\, dx.\end{aligned} \tag{3.36}$$

Multiplying (3.36) by r_ε^2 and passing to the limit, as ε goes to zero, we obtain by (3.34)

$$\int_\Omega \int_I v_1(x,y) \frac{\partial \varphi_{11}}{\partial y}(x,y)\, dydx = 0, \qquad \forall\ \varphi_{11} \in \mathcal{D}(\Omega \times I),$$

hence

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{in } \mathcal{D}'(\Omega \times I)$$

which yields, due to (3.35) and the connectedness of $\Omega \times I$,

$$v_1(x,y) = v_1(x) = u_1(x) \qquad \text{a.e. } (x,y) \in \Omega \times I.$$

By considering in (3.36) test functions $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times I, \mathbb{R}^{2\times 2}_s)$ such that $\varphi_{11} = 0$ and by multiplying (3.36) by r_ε, we get, passing to the limit, as ε goes to zero,

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial w}{\partial y} = 0 \qquad \text{in } \mathcal{D}'(\Omega \times I).$$

Then, there exists a function $g \in \mathrm{L}^2(\Omega)$ such that

$$w(x,y) = -\frac{\partial u_1}{\partial x_2}(x) y + g(x) \qquad \text{a.e. } (x,y) \in \Omega \times I.$$

Finally, we consider in (3.36) test functions $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times I, \mathbb{R}^{2\times 2}_s)$ such that, for any $(i,j) \neq (2,2)$, $\varphi_{ij} = 0$. It follows that

$$\chi_{22} = \frac{\partial w}{\partial x_2} \qquad \text{a.e. in } \Omega \times I.$$

Moereover, $\frac{\partial g}{\partial x_2} \in \mathrm{L}^2(\Omega)$, since

$$g(x) = \frac{1}{2}(w(x,y) + w(x,-y)) \quad \text{a.e. } (x,y) \in \Omega \times I \quad \text{and} \quad \frac{\partial w}{\partial x_2} \in \mathrm{L}^2(\Omega \times I).$$

We infer that

$$\frac{\partial^2 u_1}{\partial x_2^2} \in \mathrm{L}^2(\Omega) \qquad \text{and} \qquad \chi_{22}(x,y) = -\frac{\partial^2 u_1}{\partial x_2^2} y + q(x) \qquad \text{a.e. } (x,y) \in \Omega \times I,$$

where $q(x) := \frac{\partial g}{\partial x_2}(x)$ a.e. $x \in \Omega$. This proves *ii)*. □

We now proceed with the proof of Theorem 4.

Lower-bound inequality. Let $(\boldsymbol{u}^\varepsilon)$ be a sequence with bounded energy strongly converging to some $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega,\mathbb{R}^2)$. We have to prove that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(\Omega,\boldsymbol{u}^\varepsilon) \geq E_0(\boldsymbol{u}). \tag{3.37}$$

Let us first notice that the boundedness and the equicoercivity of the energy $(E_\varepsilon(\Omega,\boldsymbol{u}^\varepsilon))$ combined with the Korn inequality imply that, up to a subsequence, $(\boldsymbol{u}^\varepsilon)$ weakly converges to $\boldsymbol{u}$ in $\mathrm{H}^1_0(\Omega,\mathbb{R}^2)$. By Lemma 4 we also have

$$\frac{\partial^2 \boldsymbol{u}}{\partial x_2^2} \in \mathrm{L}^2(\Omega,\mathbb{R}^2). \tag{3.38}$$

Estimate of the energy in the matrix M_ε: Since the sequence $(E_\varepsilon(\Omega,\boldsymbol{u}^\varepsilon))$ is bounded, so is the sequence $(E^s_\varepsilon(F_\varepsilon,\boldsymbol{u}^\varepsilon))$. There exists a positive constant M such that

$$2\mu_\varepsilon \int_{F_\varepsilon} \|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2\,dx + \lambda_\varepsilon \int_{F_\varepsilon} (\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}^\varepsilon)))^2\,dx < M.$$

Assumptions (3.26) and (3.27) imply that the coefficients μ_ε and λ_ε tend to infinity as ε goes to zero. Therefore,

$$\lim_{\varepsilon\to 0}\int_{F_\varepsilon} \|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2\,dx = \lim_{\varepsilon\to 0}\int_{F_\varepsilon} (\mathrm{Tr}\, e(\boldsymbol{u}^\varepsilon))^2\,dx = 0,$$

hence,

$$\lim_{\varepsilon\to 0} E^m(F_\varepsilon,\boldsymbol{u}^\varepsilon) = \lim_{\varepsilon\to 0}\int_{F_\varepsilon}\left(2\mu\|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2 + \lambda(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}^\varepsilon)))^2\right)dx = 0.$$

Then, by the lower semicontinuity of quadratic functionals and the weak convergence of $(\boldsymbol{u}^\varepsilon)$ to $\boldsymbol{u}$ in $\mathrm{H}^1_0(\Omega,\mathbb{R}^2)$, we deduce

$$\begin{aligned}\liminf_{\varepsilon\to 0} E^m(M_\varepsilon,\boldsymbol{u}^\varepsilon) &= \liminf_{\varepsilon\to 0} E^m(\Omega,\boldsymbol{u}^\varepsilon)\\ &= \liminf_{\varepsilon\to 0}\int_\Omega \left(2\mu\|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2 + \lambda(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}^\varepsilon)))^2\right)\,dx\\ &\geq \int_\Omega \left(2\mu\|\boldsymbol{e}(\boldsymbol{u})\|^2 + \lambda(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u})))^2\right)\,dx\\ &= E^m(\Omega,\boldsymbol{u}).\end{aligned} \tag{3.39}$$

Estimate of the energy in the strips F_ε: On one hand, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} E^s_\varepsilon(F_\varepsilon,\boldsymbol{u}^\varepsilon) &= 4\liminf_{\varepsilon\to 0}\left(\frac{\mu_\varepsilon r_\varepsilon^3}{\varepsilon}\right) ⨍_{F_\varepsilon}\left(\left\|\frac{\boldsymbol{e}(\boldsymbol{u}^\varepsilon)}{r_\varepsilon}\right\|^2 + \frac{\lambda_\varepsilon}{2\mu_\varepsilon}\left[\mathrm{Tr}\left(\frac{\boldsymbol{e}(\boldsymbol{u}^\varepsilon)}{r_\varepsilon}\right)\right]^2\right)dx\\ &= 4\mu_1 \liminf_{\varepsilon\to 0} ⨍_{F_\varepsilon}\left(\left\|\frac{\boldsymbol{e}(\boldsymbol{u}^\varepsilon)}{r_\varepsilon}\right\|^2 + \frac{\lambda_\varepsilon}{2\mu_\varepsilon}\left[\mathrm{Tr}\left(\frac{\boldsymbol{e}(\boldsymbol{u}^\varepsilon)}{r_\varepsilon}\right)\right]^2\right)dx.\end{aligned}$$

On the other hand, the sequence $(\boldsymbol{u}^\varepsilon)$ has bounded energy. By Lemma 4 the sequence $(\boldsymbol{e}(\boldsymbol{u}^\varepsilon)/r_\varepsilon)$ double-scale converges to some $\boldsymbol{\chi} \in \mathrm{L}^2(\Omega\times I,\mathbb{R}^{2\times 2}_s)$. It follows that

$$\liminf_{\varepsilon\to 0} E^s_\varepsilon(F_\varepsilon,\boldsymbol{u}^\varepsilon) \geq 4\mu_1 ⨍_\Omega ⨍_I \left(\|\boldsymbol{\chi}(x,y)\|^2 + \frac{\ell}{2}\left(\mathrm{Tr}(\boldsymbol{\chi}(x,y))\right)^2\right)\,dydx.$$

Moreover, a simple comparison of quadratic forms on the set of symmetric matrices shows that, for any $\boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}_s$, one has

$$\|\boldsymbol{\sigma}\|^2 + \frac{\ell}{2}\left(\mathrm{Tr}(\boldsymbol{\sigma})\right)^2 \geq 2\left(\frac{\ell+1}{\ell+2}\right)\sigma_{22}^2.$$

This inequality combined with Lemma 4 yields

$$\begin{aligned}
\liminf_{\varepsilon\to 0} E^s_\varepsilon(F_\varepsilon, \boldsymbol{u}^\varepsilon) &\geq 4\mu_1 \fint_\Omega \fint_I 2\left(\frac{\ell+1}{\ell+2}\right)\left(\chi_{22}(x,y)\right)^2 dydx \\
&\geq 3k \fint_\Omega \fint_I \left(q(x) - \frac{\partial^2 u_1}{\partial x_2^2}(x) y\right)^2 dydx \\
&\geq k \fint_\Omega \left(\frac{\partial^2 u_1}{\partial x_2^2}(x)\right)^2 dx. \qquad (3.40)
\end{aligned}$$

Boundary conditions: Consider the extended domain $\tilde{\Omega} := (0,1)\times(-1,2)$ where $\tilde{\boldsymbol{u}}^\varepsilon$ and $\tilde{\boldsymbol{u}}$ are defined by:

$$\begin{cases} \tilde{\boldsymbol{u}}^\varepsilon(x) := \boldsymbol{u}^\varepsilon(x) & \text{if} \quad x \in \Omega, \\ \tilde{\boldsymbol{u}}^\varepsilon(x) := 0 & \text{if} \quad x \in \tilde{\Omega}\setminus\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{\boldsymbol{u}}(x) := \boldsymbol{u}(x) & \text{if} \quad x \in \Omega, \\ \tilde{\boldsymbol{u}}(x) := 0 & \text{if} \quad x \in \tilde{\Omega}\setminus\Omega. \end{cases}$$

The energy $E_\varepsilon(\tilde{\Omega}, \tilde{\boldsymbol{u}}^\varepsilon)$ is bounded and $(\tilde{\boldsymbol{u}}^\varepsilon)$ strongly converges to $\tilde{\boldsymbol{u}}$ in $\mathrm{L}^2(\tilde{\Omega}, \mathbb{R}^2)$. Lemma 4 applied to $\tilde{\boldsymbol{u}}^\varepsilon$ yields

$$\tilde{\boldsymbol{u}} \in \mathrm{H}^1_0(\tilde{\Omega}, \mathbb{R}^2) \quad \text{and} \quad \frac{\partial^2 \tilde{u}_1}{\partial x_2^2} \in \mathrm{L}^2(\tilde{\Omega}, \mathbb{R}^2).$$

This implies that

$$\frac{\partial u_1}{\partial x_2}(x) = 0 \quad \text{a.e.} \;\; x \in (0,1)\times\{0,1\}. \qquad (3.41)$$

The lower-bound inequality is then proved by putting (3.41) together with (3.40), (3.39) and (3.38).

Upper-bound inequality. Let $\boldsymbol{u} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$ such that $E_0(\boldsymbol{u}) < \infty$. We have to construct an approximating sequence $(\boldsymbol{u}^\varepsilon)$ strongly converging to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$ and satisfying

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\Omega, \boldsymbol{u}^\varepsilon) \leq E_0(\boldsymbol{u}). \qquad (3.42)$$

Using a regularization by convolution, we can show that $\mathcal{V} \cap \mathcal{D}(\Omega, \mathbb{R}^2)$ is dense in the space $\mathcal{V}$ defined by (3.29) endowed with the norm defined by (3.33). Therefore, since the energy E_0, defined by(3.28), is clearly continuous with respect to this norm, we can restrict ourselves to prove the upper-bound inequality for a function $\boldsymbol{u} \in \mathcal{V} \cap \mathcal{D}(\Omega, \mathbb{R}^2)$.

Construction of the approximating sequence: Let (R_ε) be a sequence of real numbers satisfying $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. We define the transition layer by

$$C_\varepsilon := \{x \in \Omega : \; 1 < |y_\varepsilon(x)| < (R_\varepsilon/r_\varepsilon)\}.$$

This is a subset of the matrix M_ε. The remaining part of M_ε is denoted by

$$B_\varepsilon := \{x \in \Omega : \; |y_\varepsilon(x)| > (R_\varepsilon/r_\varepsilon)\}.$$

In each period P^i_ε we set

$$C^i_\varepsilon := C_\varepsilon \cap P^i_\varepsilon \qquad \text{and} \qquad B^i_\varepsilon := B_\varepsilon \cap P^i_\varepsilon.$$

For any $i \in \{1, \cdots, \varepsilon^{-1}\}$, $x_2 \in (0,1)$ and $y \in \mathbb{R}$, we define

$$\boldsymbol{v}^{\varepsilon,i}(x_2) := \fint_I \boldsymbol{u}(x^i_\varepsilon + r_\varepsilon y, x_2)\, dy, \tag{3.43}$$

$$w^{\varepsilon,i}_1(x_2, y) := v^{\varepsilon,i}_1(x_2) + \frac{\ell r^2_\varepsilon}{2(\ell+2)} \frac{\partial^2 v^{\varepsilon,i}_1}{\partial x^2_2}(x_2) y^2 \qquad \text{and} \qquad w^{\varepsilon,i}_2(x_2, y) := -r_\varepsilon \frac{\partial v^{\varepsilon,i}_1}{\partial x_2}(x_2) y.$$

Since $\boldsymbol{u} \in \mathcal{D}(\Omega, \mathbb{R}^2)$, both functions $\boldsymbol{v}^{\varepsilon,i}$ and $\boldsymbol{w}^{\varepsilon,i} := (w^{\varepsilon,i}_1, w^{\varepsilon,i}_2)$ are infinitely differentiable in $(0,1)$ and $(0,1)\times\mathbb{R}$, respectively. Finally, the approximating sequence $(\boldsymbol{u}^\varepsilon)$ is defined as follows:

$$\boldsymbol{u}^\varepsilon(x) := \begin{cases} \boldsymbol{u}(x) & \text{if} \quad x \in B_\varepsilon \\ \boldsymbol{w}^{\varepsilon,i}(x_2, y_\varepsilon(x)) & \text{if} \quad x \in F^i_\varepsilon \\ \gamma_{\varepsilon,i}(x_1)\boldsymbol{w}^{\varepsilon,i}(x_2, y_\varepsilon(x)) + (1-\gamma_{\varepsilon,i}(x_1))\boldsymbol{u}(x) & \text{if} \quad x \in C^i_\varepsilon, \end{cases} \tag{3.44}$$

where $\gamma_{\varepsilon,i} : (0,1) \to [0,1]$ is the continuous interpolation function

$$\gamma_{\varepsilon,i}(t) := \begin{cases} \dfrac{t - (x^i_\varepsilon - R_\varepsilon)}{R_\varepsilon - r_\varepsilon} & \text{if} \quad x^i_\varepsilon - R_\varepsilon \le t \le x^i_\varepsilon - r_\varepsilon, \\ 1 & \text{if} \quad x^i_\varepsilon - r_\varepsilon \le t \le x^i_\varepsilon + r_\varepsilon, \\ \dfrac{(x^i_\varepsilon + R_\varepsilon) - t}{R_\varepsilon - r_\varepsilon} & \text{if} \quad x^i_\varepsilon + r_\varepsilon \le t \le x^i_\varepsilon + R_\varepsilon, \end{cases} \tag{3.45}$$

for any $i \in \{1, \cdots, \varepsilon^{-1}\}$. It is easy to see that $\boldsymbol{u}^\varepsilon$ belongs to $\mathrm{H}^1_0(\Omega, \mathbb{R}^2)$. Moreover, the sequence $(\boldsymbol{u}^\varepsilon)$ strongly converges to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$. Indeed, $\boldsymbol{u}^\varepsilon = \boldsymbol{u}$ in B_ε, the measure $|\Omega \setminus B_\varepsilon|$ goes to zero with ε and $(\boldsymbol{u}^\varepsilon)$ is uniformly bounded in $F_\varepsilon \cup C_\varepsilon$. This yields the strong convergence of $(\boldsymbol{u}^\varepsilon)$ to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega, \mathbb{R}^2)$.

Now, we have to estimate the energy of $\boldsymbol{u}^\varepsilon$ in the different zones of the domain: The matrix zone B_ε, the reinforced zone F_ε and the transition layer C_ε.

Estimate of the energy of $\boldsymbol{u}^\varepsilon$ in the matrix B_ε:
For any $x \in B_\varepsilon$ we have $\boldsymbol{u}^\varepsilon(x) = \boldsymbol{u}(x)$. Then $E^m(B_\varepsilon, \boldsymbol{u}^\varepsilon) = E^m(B_\varepsilon, \boldsymbol{u})$. Since $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega, \mathbb{R}^2)$ and $|\Omega \setminus B_\varepsilon|$ tends to zero, so does $E^m(\Omega \setminus B_\varepsilon, \boldsymbol{u})$. It follows that

$$\lim_{\varepsilon \to 0} E^m(B_\varepsilon, \boldsymbol{u}^\varepsilon) = E^m(\Omega, \boldsymbol{u}). \tag{3.46}$$

Estimate of the energy of $\boldsymbol{u}^\varepsilon$ in the reinforced zone F_ε:
We begin by estimating the energy of $\boldsymbol{u}^\varepsilon$ in one strip F^i_ε. Using the definitions (3.25) and (3.44) we obtain, for any $x = (x_1, x_2) \in \Omega$,

$$\begin{cases} \boldsymbol{e}_{11}(\boldsymbol{u}^\varepsilon)(x) = \dfrac{\ell r_\varepsilon}{\ell+2} \dfrac{\partial^2 v^{\varepsilon,i}_1}{\partial x^2_2}(x_2) y_\varepsilon(x), \\ \boldsymbol{e}_{22}(\boldsymbol{u}^\varepsilon)(x) = -r_\varepsilon \dfrac{\partial^2 v^{\varepsilon,i}_1}{\partial x^2_2}(x_2) y_\varepsilon(x), \\ \boldsymbol{e}_{12}(\boldsymbol{u}^\varepsilon)(x) = \dfrac{\ell r^2_\varepsilon}{4(\ell+2)} \dfrac{\partial^3 v^{\varepsilon,i}_1}{\partial x^3_2}(x_2)(y_\varepsilon(x))^2. \end{cases} \tag{3.47}$$

The energy of $\boldsymbol{u}^\varepsilon$ in one strip F^i_ε is given by

$$E^s_\varepsilon(F^i_\varepsilon, \boldsymbol{u}^\varepsilon) = \int_{F^i_\varepsilon} \left(2\mu_\varepsilon \|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2 + \lambda_\varepsilon \left(\mathrm{Tr}(\boldsymbol{e}(\boldsymbol{u}^\varepsilon))\right)^2\right)\, dx$$
$$= \int_{F^i_\varepsilon} \left[2\mu_\varepsilon\left((\boldsymbol{e}_{11}(\boldsymbol{u}^\varepsilon))^2 + (\boldsymbol{e}_{22}(\boldsymbol{u}^\varepsilon))^2 + 2(\boldsymbol{e}_{12}(\boldsymbol{u}^\varepsilon))^2\right) + \lambda_\varepsilon \left(\boldsymbol{e}_{11}(\boldsymbol{u}^\varepsilon) + \boldsymbol{e}_{22}(\boldsymbol{u}^\varepsilon)\right)^2\right] dx.$$

By (3.47) this energy writes

$$\begin{aligned} E^s_\varepsilon(F^i_\varepsilon, \boldsymbol{u}^\varepsilon) &= \int_{F^i_\varepsilon} r^2_\varepsilon \left[2\mu_\varepsilon\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \lambda_\varepsilon \frac{4}{(\ell+2)^2}\right] \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}(x_2) y_\varepsilon(x)\right)^2 dx \\ &+ \int_{F^i_\varepsilon} \frac{\mu_\varepsilon r_\varepsilon^4 \ell^2}{4(\ell+2)^2} \left(\frac{\partial^3 v_1^{\varepsilon,i}}{\partial x_2^3}(x_2)(y_\varepsilon(x))^2\right)^2 dx. \end{aligned} \tag{3.48}$$

Integrating $y_\varepsilon(x)$ with respect to the variable x_1, the first term of (3.48) can be then rewritten

$$\mu_\varepsilon r^2_\varepsilon \left[2\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \frac{\lambda_\varepsilon}{\mu_\varepsilon} \frac{4}{(\ell+2)^2}\right] \int_{F^i_\varepsilon} \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}(x_2) y_\varepsilon(x)\right)^2 dx$$
$$= \frac{2}{3}\mu_\varepsilon r^3_\varepsilon \left[2\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \frac{\lambda_\varepsilon}{\mu_\varepsilon} \frac{4}{(\ell+2)^2}\right] \int_0^1 \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}\right)^2 dx_2.$$

Summing over $i \in \{1, \cdots, \varepsilon^{-1}\}$, we obtain that the part of $E^s_\varepsilon(F_\varepsilon, \boldsymbol{u}^\varepsilon)$ corresponding to the first term of (3.48) writes

$$\begin{aligned} &\int_{F_\varepsilon} \left[2\mu_\varepsilon\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \lambda_\varepsilon \frac{4}{(\ell+2)^2}\right] r^2_\varepsilon \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}(x_2) y_\varepsilon(x)\right)^2 dx \\ &= \frac{2}{3}\mu_\varepsilon r^3_\varepsilon \left[2\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \frac{\lambda_\varepsilon}{\mu_\varepsilon} \frac{4}{(\ell+2)^2}\right] \sum_{i=1}^{\varepsilon^{-1}} \int_0^1 \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}\right)^2 dx_2. \end{aligned} \tag{3.49}$$

Taking into account the assumptions (3.26) and (3.27), we easily obtain

$$\lim_{\varepsilon\to 0} \left(\frac{2}{3}\frac{\mu_\varepsilon r^3_\varepsilon}{\varepsilon} \left[2\left(\frac{\ell^2}{(\ell+2)^2} + 1\right) + \frac{\lambda_\varepsilon}{\mu_\varepsilon} \frac{4}{(\ell+2)^2}\right]\right) = k, \tag{3.50}$$

where k is defined by (3.30). Recalling the definition (3.43) of $\boldsymbol{v}^{\varepsilon,i}$ and using Jensen's inequality we obtain

$$\varepsilon \sum_{i=1}^{\varepsilon^{-1}} \int_0^1 \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}\right)^2 dx_2 \leq \fint_{F_\varepsilon} \left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2 dx. \tag{3.51}$$

Similarly, we get

$$\varepsilon \sum_{i=1}^{\varepsilon^{-1}} \int_0^1 \left(\frac{\partial^3 v_1^{\varepsilon,i}}{\partial x_2^3}\right)^2 dx_2 \leq \fint_{F_\varepsilon} \left(\frac{\partial^3 u_1}{\partial x_2^3}\right)^2 dx.$$

Note that the term in $E^s_\varepsilon(F_\varepsilon, \boldsymbol{u}^\varepsilon)$ involving third derivatives of $\boldsymbol{v}^{\varepsilon,i}$ is of order $O(r^2_\varepsilon)$ and thus goes to zero with ε. From (3.49), (3.50), the inequality (3.51) and Lemma 3, we deduce

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} E^s_\varepsilon(F_\varepsilon,\boldsymbol{u}^\varepsilon) &\le k \limsup_{\varepsilon\to 0} \sum_{i=1}^{\varepsilon^{-1}} \int_0^1 \left(\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}\right)^2 dx_2 \\
&\le k \limsup_{\varepsilon\to 0} \fint_{F_\varepsilon} \left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2 dx \\
&\le k \fint_{\Omega} \left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2 dx.
\end{aligned} \tag{3.52}
$$

Estimate of the energy of $\boldsymbol{u}^\varepsilon$ in the transition layer C_ε:
By the definition (3.44) of $\boldsymbol{u}^\varepsilon$ in C^i_ε, we have for any $x \in C^i_\varepsilon$,

$$
\begin{aligned}
\frac{\partial u_1^\varepsilon}{\partial x_1}(x) &= \gamma'_{\varepsilon,i}(x_1)\left(w_1^{\varepsilon,i}(x_2, y_\varepsilon(x)) - u_1(x)\right) + \gamma_{\varepsilon,i}(x_1)\frac{r_\varepsilon \ell}{\ell+2}\frac{\partial^2 v_1^{\varepsilon,i}}{\partial x_2^2}(x_2) y_\varepsilon(x) \\
&\quad + (1-\gamma_{\varepsilon,i}(x_1))\frac{\partial u_1}{\partial x_1}(x).
\end{aligned} \tag{3.53}
$$

Taking into account the definitions (3.45), (3.43) of $\gamma_{\varepsilon,i}$ and $\boldsymbol{v}^{\varepsilon,i}$, respectively, there exists a positive constant c such that

$$
\begin{aligned}
|w_1^{\varepsilon,i}(x_2, y_\varepsilon(x)) - u_1(x)| &\le \left| \fint_I (u_1(x^i_\varepsilon + r_\varepsilon y, x_2) - u_1(x_1,x_2))\, dy \right| + c\, R_\varepsilon^2 \\
&\le c\,(r_\varepsilon + R_\varepsilon + R_\varepsilon^2).
\end{aligned} \tag{3.54}
$$

The last two terms of (3.53) are obviously bounded and $|\gamma'(t)| \le (R_\varepsilon - r_\varepsilon)^{-1}$ a.e. $t \in (0,1)$. It follows from (3.53) and (3.54) that the first partial derivative of u_1^ε with respect to x_1 is bounded in C_ε. We similarly prove the boundedness of the other first partial derivatives of $\boldsymbol{u}^\varepsilon$ in C_ε. Moreover, owing to (3.24), $|C_\varepsilon| = O((R_\varepsilon - r_\varepsilon)/\varepsilon)$ and tends to zero, hence

$$
\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \|\boldsymbol{e}(\boldsymbol{u}^\varepsilon)\|^2\, dx = 0.
$$

This implies that

$$
\lim_{\varepsilon\to 0} E^m(C_\varepsilon, \boldsymbol{u}^\varepsilon) = 0. \tag{3.55}
$$

The upper-bound inequality is a consequence of the strong convergence of $(\boldsymbol{u}^\varepsilon)$ to $\boldsymbol{u}$ in $\mathrm{L}^2(\Omega,\mathbb{R}^2)$, combined with the estimates (3.46), (3.52) and (3.55). Theorem 4 is thus proved. □

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